MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

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STATISTICAL DECISION PROCEDURES IN INDUSTRY

III. ACCEPTANCE SAMPLING BY ATTRIBUTES

JOHN M. HOWELL, Los Angeles City College

3.1. Introduction. In the first two articles of this series, the problem of control of a process was discussed. Here the situation considered was one in which the producer had full control of the process which made the parts. We will now turn our attention to the consumer or the company receiving parts which have been made by some other company. Here the problem is somewhat different since the consumer does not have full control of the manufacturing process.

Incoming material is often examined at a point in the plant designated as "Receiving Inspection." The basic tools used here are generally called "Acceptance Sampling." When receiving material which may be raw material, component parts or assemblies, one of the first questions which arises is: "Shall we look at all of the items or only a portion of them?" This question is nonsensical if the test is destructive, so we shall confine our attention in this section to non-destructive tests.

We should examine all of the product when one of the following conditions exists:

1) Occurrence of a defect will cause loss of life or serious casualty to personnel.
2) A defect will certainly cause serious malfunction of equipment.

Inspection of all items of a product is ordinarily called "100% inspection" which gives the idea that the "outgoing" quality is perfect, but sometimes this is not the case.

Also we may wish to inspect all units of a product under certain conditions:

1) Lot size is small.
2) Poor or unknown incoming quality.

On the other hand, we may wish to inspect only a portion of the product when:

1) A defect will not cause serious accident or malfunction.
2) Parts are used in groups.
3) Defective parts will be detected at a later stage of assembly or manufacture.

If it is decided to inspect only a portion of the product, the next question which arises is "How much should be inspected?"

A so-called "plan" which has had rather widespread use in certain places is as follows:

1) Inspect 10% of the number of units presented.
2) If no defects are found, accept all of the product.
3) If one or more defects are found, reject the lot.

However, this so-called plan has several shortcomings:

1) Accepted material is very nearly same quality as that presented.
2) Accepted material gets poorer as the material presented gets poorer.
3) Not enough pieces of small lots are inspected and too many of larger lots are inspected.

4) The plan has no objective. We did not state what was desired before we started sampling.

About the time that Shewhart was developing control techniques, Dodge and Romig [3.1] were developing acceptance procedures. The discussion herein in general follows the Dodge-Romig average quality protection plans.

3.2. Procedure for Single Sampling. We shall start our discussion with the simple case of lots of constant size N, and use single sampling. That is, from each lot, a sample of size n is selected at random. If the number of defective pieces, x, found in the sample is less than or equal to an acceptance number, c, the entire lot is accepted. If the lot is not accepted, the balance of the lot will be "sorted," that is, all pieces are inspected and the defective pieces removed. This "sorting" is necessary if we desire the "outgoing quality" to be better than the "incoming quality." These last two terms are with respect to the inspection process, as illustrated in Fig. 3.1.

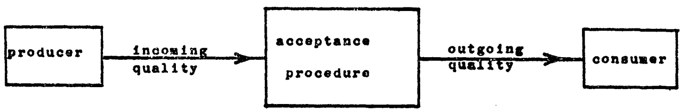

FIG. 3.1. Acceptance sampling flow chart.

In practice, the sampling may be done by the producer or the consumer or an independent agency, and the sorting may not be done by the same company or department as the sampling. However, we shall pay more attention here to theoretical considerations than to practical ones.

3.3. Theory Underlying Sampling Procedures. The probability of finding x defective items in a sample of n pieces taken from a lot of N pieces which is p fraction defective is given by the hypergeometric distribution:

$$P(x; n, N, p') = \frac{\binom{p'N}{x}\binom{N - p'N}{n - x}}{\binom{N}{n}} \tag{3.1}$$

However when the sample size is less than one tenth of the lot size, this probability is approximated by the binomial distribution:

$$P(x; n, p') = \binom{n}{d} p'^{x}(1 - p')^{n-x} \tag{3.2}$$

Further, when the fraction defective is less than one tenth, this is approximated by the Poisson distribution:

$$P(x; p'n) = \frac{(p'n)^{x}e^{-p'n}}{x!} \tag{3.3}$$

Since in practice the above two conditions are usually met, the Poisson distribution is generally used for acceptance sampling problems. The probability of acceptance under these conditions will thus be:

$$P_a = \sum_{x=0}^{c} P(x; p'n) \tag{3.4}$$

A very brief table of the cumulative Poisson distribution is given below. A more complete table is given in [2.5].

TABLE 3.1

CUMULATIVE POISSON DISTRIBUTION

		$p'n$									
		0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	0	.607	.368	.223	.135	.082	.050	.040	.018	.011	.007
	1	.910	.736	.558	.406	.287	.199	.136	.092	.061	.043
	2	.986	.920	.809	.677	.544	.423	.321	.238	.174	.125
	3	.998	.981	.934	.857	.758	.647	.537	.433	.342	.265
	4	1.000	.996	.981	.947	.891	.815	.725	.629	.532	.450
c	5		.999	.996	.983	.958	.916	.858	.785	.703	.616
	6		1.000	.999	.995	.986	.966	.935	.889	.831	.762
	7			1.000	.999	.996	.988	.973	.949	.913	.867
	8				1.000	.999	.996	.990	.979	.960	.932
	9					1.000	.999	.997	.992	.983	.968
	10						1.000	.999	.997	.993	.986
	11							1.000	.999	.998	.994
	12								1.000	.999	.998
	13									1.000	.999
	14										1.000

TABLE 3.2

COMPARISON OF ACTUAL AND THEORETICAL VALUES

d	Samples 1–20				Samples 21–30				Samples 31–40			
	Freq.	Cum.	Act.	Theo.	Freq.	Cum.	Act.	Theo.	Freq.	Cum.	Act.	Theo.
0	4	4	.20	.135		0	0	.011	4	4	.4	.368
1	7	11	.55	.406		0	0	.061	4	8	.8	.736
2	4	15	.75	.677	1	1	.1	.174	1	9	.9	.920
3	2	17	.85	.857	2	3	.3	.342	1	10	1.0	.981
4	2	19	.95	.947	3	6	.6	.532				.996
5	1	20	1.00	.983	1	7	.7	.703				.999
6				.995	2	9	.9	.831				1.000
7				.999		9	.9	.913				
8				1.000		9	.9	.960				
9					1	10	1.0	.983				

3.4. Comparison of Actual and Theoretical Values. A comparison of results of an experiment with theoretical predictions is always of some interest. Suppose that we review the experiment which was reported in Table 2.1 and 2.2. For samples 1–20, the population fraction defective was .04, so that $p'n=(.04)(50)=2$. Similarly for samples 21–30, $p'n=(.09)(50)=4.5$, and for samples 31–40, $p'n=(.02)(50)=1$. The results of this experiment are tabulated in Table 3.2 and compared with theoretical values.

Considering the small number of samples, there is a good agreement between actual and theoretical values.

3.5. Curves for Sampling Plans. In order to understand the operation of sampling plans, it is necessary to draw certain curves:

1) *OC.*—Operating characteristic curve. Relation between incoming percent defective and probability of acceptance of lot.
2) *ATI.*—Average total inspected. Relation between incoming percent defective and average total number inspected per lot including sampling and "culling."
3) *AOQ.*—Average outgoing quality. Relation between incoming percent defective and average outgoing quality.

The *OC* curve is found by assuming various values of incoming percent, p', defective, determining the average number of defective pieces per sample, pn, and then determining the probability of acceptance, P_a, from the Poisson p distribution if $p'<.1$.

The *ATI* values are found from the following:

$$ATI = P_a n + (1 - P_a)N. \tag{3.5}$$

That is, if the lot is accepted, we look at only n pieces, but if the lot is not accepted, we look at the entire lot.

The *AOQ* values are found as follows:

$$AOQ = \frac{p'P_a(N - n)}{N}. \tag{3.6}$$

The defective pieces are found only in the uninspected portion of accepted lots, so that the number involved is $N-n$. The probability that a lot is accepted is P_a and the probability of finding a defect is p', so the number of defective pieces is the product of these. The *AOQ* is then found by dividing this value by the size of the lot. This formula assumes that all defects found are repaired or replaced by good pieces. When the sample size is small compared to the lot size, pP_a is a good approximation to the outgoing quality.

If the defective pieces found are not repaired, or replaced, then this formula must be modified to:

$$AOQ = \frac{p'P_a(N - n)}{N - p'[P_a n + (1 - P_a)N]}. \tag{3.7}$$

This calculation is not usually made and will not be greatly different from that given above when p is small.

This curve will reach a maximum value and then recede because as the quality of incoming product becomes poorer, fewer lots will be accepted. When fewer lots are accepted, more will be sorted and thus the quality improved.

The highest point which this curve reaches is called the *AOQL* or Average Outgoing Quality Limit.

3.6. An Example. Suppose that lots of 1000 pieces were received, a sample of 50 was taken from each and an acceptance number of 2 was used. By assuming values of incoming quality, we can calculate:

1) Probability of accepting a lot.
2) Average total inspected.
3) Average outgoing quality.

This is done below and the curves drawn. From these curves, several points should be apparent.

1) If incoming quality is good, most lots will be accepted.
2) If incoming quality is good, sampling is efficient. That is, not much sorting must be done.
3) Regardless of incoming quality, the outgoing quality will be less than about 2.6% on the average.

TABLE 3.3

CALCULATION OF CURVES FOR $N=1000$, $n=50$, $c=2$

Inc Qual p'	Exp Def $p'n$	Prob. of Acc. Lot P_a	*ATI* $P_a n+(1-P_a)N$	*AOQ* $\frac{p'P_a(N-n)}{N}$
0	0	1.000	50	0
.01	.5	.986	64	.0094
.02	1.0	.920	126	.0175
.03	1.5	.809	231	.0231
.04	2.0	.677	357	.0257
.05	2.5	.544	483	.0258
.06	3.0	.423	598	.0241
.07	3.5	.321	695	.0213
.08	4.0	.238	774	.0181
.09	4.5	.174	835	.0149
.10	5.0	.125	881	.0119
1.00	50	0	1000	0

3.7. The Average Outgoing Quality Limit, AOQL. It will be noted that the average outgoing quality curve has a maximum value called the Average Outgoing Quality Limit, often denoted by *AOQL* or p_L. This is a limiting value for the outgoing quality. Regardless of the value of incoming quality, the outgoing quality will be no worse than the *AOQL* on the average. Again let us point out the necessity for several lots for this to hold.

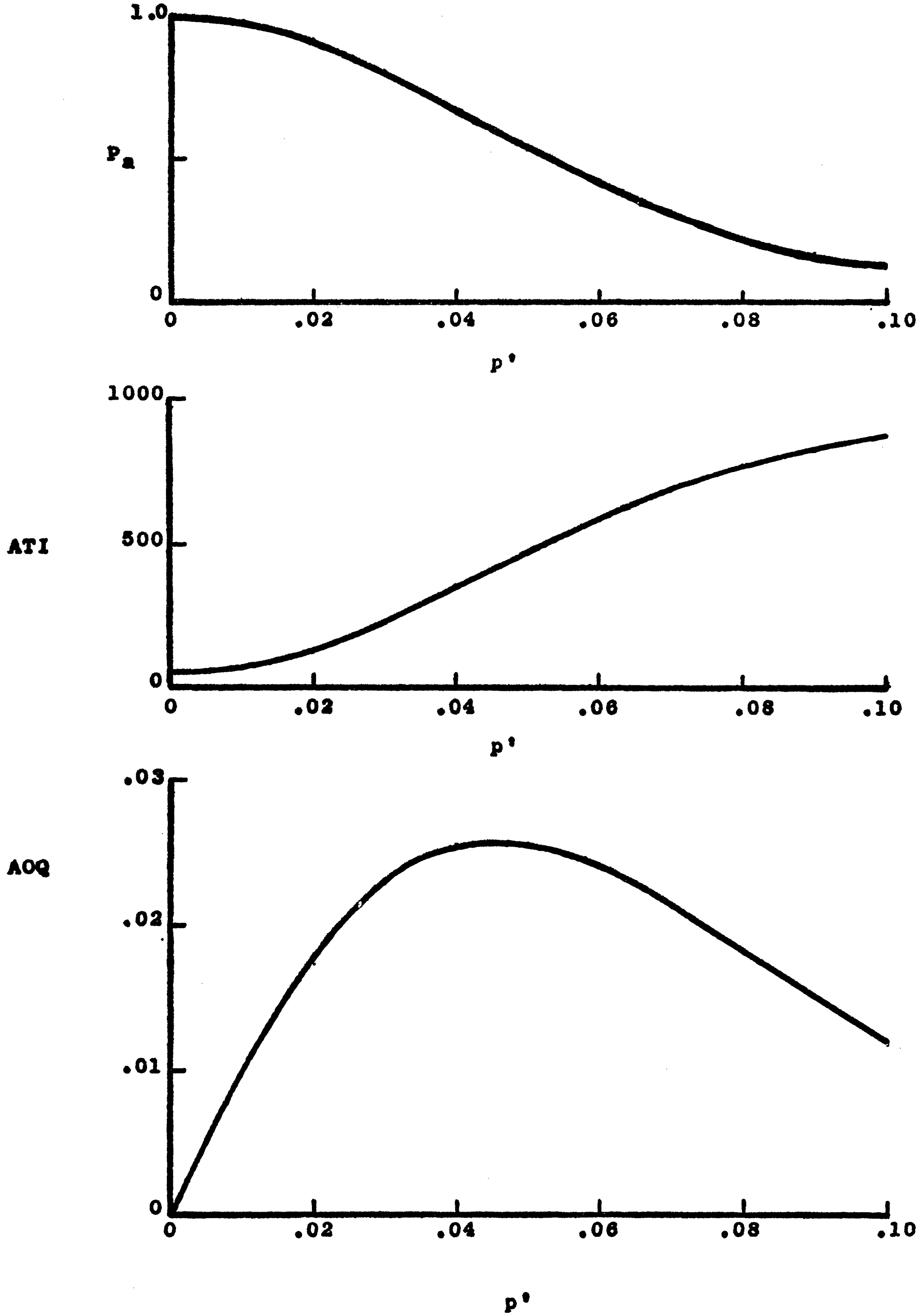

FIG. 3.2. Curves for $N=1000$, $n=50$, $n=2$.

The $AOQL$ can be calculated by a formula and constants given in [3.1].

C	0	1	2	3	4
y	.3679	.8400	1.371	1.942	2.544

$$p_L = y\left(\frac{1}{n} - \frac{1}{N}\right). \tag{3.8}$$

For the example given above,

$$p_L = 1.371\left(\frac{1}{50} - \frac{1}{1000}\right) = .0260,$$

which agrees with the values found when plotting the curve.

By solving (3.7) for n, we have a method of finding the sample size to achieve a desired result. Suppose that the consumer who is going to receive lots of size 1000 desires to maintain a quality level of 1%,

$$n = \frac{yN}{p_L N + y}, \tag{3.9}$$

Substituting values in (3.8) we obtain the following pairs of values which will maintain the desired quality.

c	0	1	2	3	4
n	35	80	120	160	200

The sample sizes here were rounded off to the closest 5 pieces.

We now have several sampling plans, all of which will maintain a 1% quality level. Which one should we use? If we are interested in minimum amount of total inspection, we should examine the ATI charts. Calculations for these charts are given below, and an expanded chart of a portion of the ATI curve is given. From this we see that the $n=35$ plan is best from 0 to about .15%, then the $n=80$ plan is lowest to about .55%, and higher than that, the $n=120$ plan is lowest.

Thus in order to pick the most efficient plan we need to know the incoming quality. Of course, if we really knew the incoming quality, we would not need to sample, so we generally find a "process average" which is the average quality of the last five or ten samples.

Comparison with values in Dodge-Roming tables show:

Process Average	Sampling Plan	
	n	c
0 to .20%	35	0
.21 to .60%	80	1
.61 to 1.00%	120	2

TABLE 3.4

CALCULATION OF CURVES FOR $AOQL=1\%$ PLANS

$p'\%$	$N=1000$, $n=35$, $c=0$				$N=1000$, $n=80$, $c=1$				$N=1000$, $n=120$, $c=2$			
	$p'n$	P_a	ATI	AOQ %	$p'n$	P_a	ATI	AOQ %	$p'n$	P_a	ATI	AOQ %
	0	1.000	35	0	0	1.000	80	0	0	1.000	120	0
.2	.07	.932	101	.18	.16	.989	90	.18	.24	.998	122	.18
.4	.14	.869	161	.33	.32	.958	118	.35	.48	.987	131	.35
.6	.21	.811	217	.47	.48	.916	157	.50	.72	.965	151	.51
.8	.28	.756	270	.58	.64	.864	205	.64	.96	.927	184	.65
1	.35	.705	320	.68	.8	.809	256	.74	1.2	.879	227	.78
2	.70	.497	524	.96	1.6	.525	517	.97	2.4	.570	498	1.00
3	1.05	.351	662	1.01	2.4	.302	722	.83	3.6	.303	733	.80
4	1.40	.247	762	.95	3.2	.171	843	.63	4.8	.143	874	.50
5	1.75	.176	830	.85	4.0	.092	915	.42	6.0	.062	945	.27
6	2.10	.122	882	.71	4.8	.048	956	.26	7.2	.025	978	.13
7	2.45	.086	917	.58	5.6	.024	978	.15	8.4	.010	991	.06
8	2.80	.061	941	.47	6.4	.012	989	.09	9.6	.004	996	.03
9	3.15	.043	958	.37	7.2	.006	994	.05	10.8	.002	998	.02
10	3.50	.030	971	.29	8.0	.003	997	.03	12.0	.001	999	.01

3.8. Some Pitfalls in Acceptance Sampling. People often wonder why sorting is necessary. Why can't we merely accept or reject the lot? Consider the following situation. A producer is producing product at a quality level which will be accepted half of the time by the consumers sampling plan. He presents 1024 lots to the consumer and the consumer accepts 512, returning 512. The producer knows that the consumer is using a sampling plan and he returns the 512 lots, and 256 are accepted. The reader should see that the consumer will eventually accept all the lots. Lots which are accepted only half of the time are usually much poorer than desired, so the accepted material is of low quality.

Beginners using sampling plans often forget that the results are guaranteed "on the average." Remember that a sample size of 50 and an accepted number of 2 was associated with an $AOQL$ of 2.6%. So our neophyte examines a sample of 50, finds 2 defects and says, "This lot has no more than 2.6% defective items in it." If one consults a table of confidence limits for the binomial distribution, he would find that 95% confidence limits for this situation are about 0.5 to 13.7% and 99% confidence limits are 0.2 to 17.2%. The percent defective of *one* lot may be considerably larger than the $AOQL$, but if we had many (say 50) lots we would be reasonably sure that the *average* quality was less than the $AOQL$.

3.9. Other Sampling Plans. The Dodge-Romig tables include double as well as single sampling and lot quality protection plans in addition to the average quality protection plans.

Tables [3.2] developed for use on government contracts are based on the same theory as the Dodge-Romig tables, but are considerably different in ad-

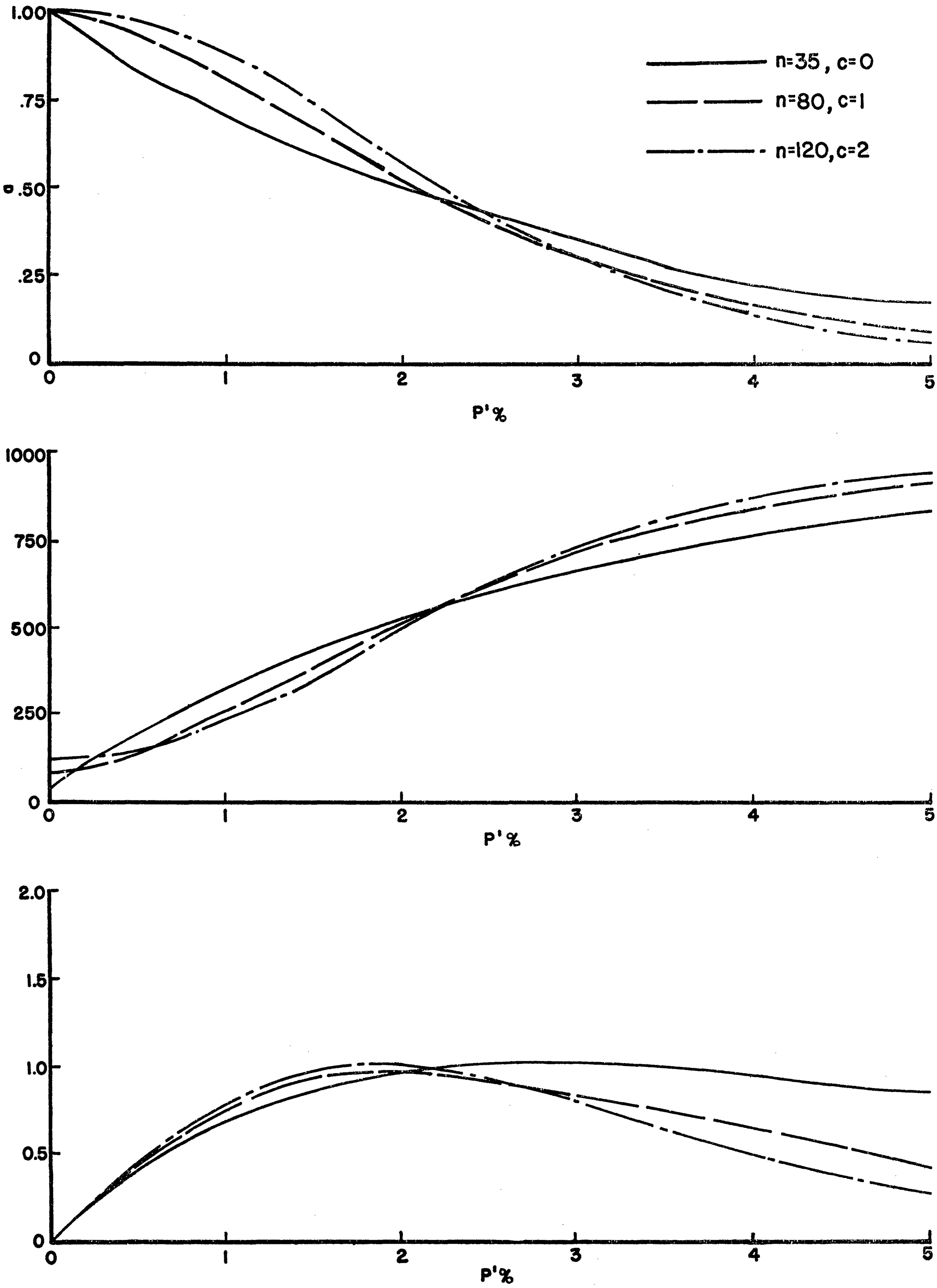

FIG. 3.3. Curves for $AOQL = 1\%$ plans.

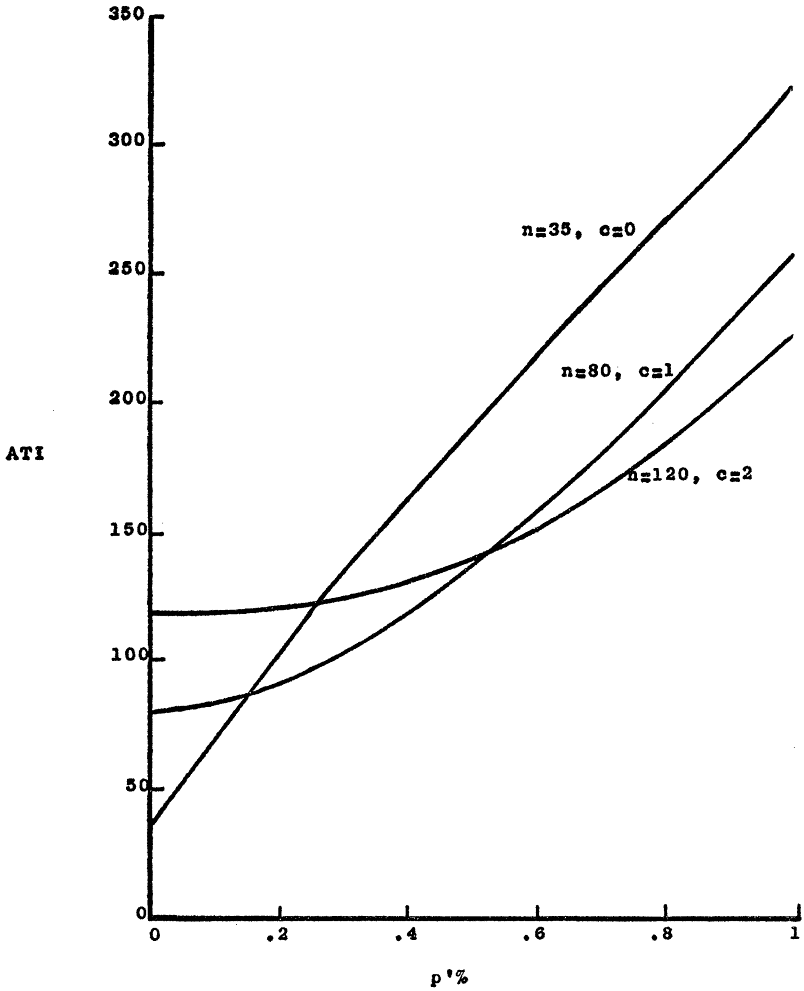

FIG. 3.4. *ATI* for *AOQL*=1% plans.

ministration. These tables include single, double and multiple sampling and also defects per unit sampling.

References

3.1 Harold F. Dodge and Harry G. Romig, Sampling Inspection Tables, John Wiley & Sons, 1959.

3.2. MIL-STD-105B, Sampling Procedures and Tables for Inspection by Attributes, Superintendent of Documents, Washington, D. C.

KHAYYAM'S SOLUTION OF CUBIC EQUATIONS

A. R. AMIR-MOÉZ, University of Florida

Omar Khayyam (1044–1123), Persian philosopher and mathematician, had a very interesting geometric solution of the third degree equations. We shall study Khayyam's method and its extension to solving fourth degree equations. This extension is due to M. Hachtroudi, Professor of Mathematics at the University of Teheran. The key to Khayyam's method is the following:

1. THEOREM: *The four points of intersection of two parabolas whose axes are perpendicular are on a circle.*

Proof: For convenience, without loss of generality, we choose the parabolas

$$y^2 = 4p(x - a), \qquad x^2 = 4q(y - b).$$

It is clear that the axes of these parabolas are perpendicular to one another. Now if we add these equations we get

$$x^2 + y^2 - 4px - 4qy + 4ap + 4bq = 0$$

which is a circle with center $(2p, 2q)$. This proves the theorem. Here the complex points of intersection have also been considered. Omar proved this theorem synthetically. We shall leave that to the reader as an exercise.

2. Solution of cubic equations: Any third degree equation can be written as

$$x^3 + lx^2 + mx + n = 0. \tag{2.1}$$

If we discuss the solution of a fourth degree equation such as

$$z^4 + az^3 + bz^2 + cz + d = 0, \tag{2.2}$$

then (2.1) will be a special case of (2.2). That is, we consider

$$x^4 + lx^3 + mx^2 + nx = 0. \tag{2.3}$$

Then we ignore the root $x=0$, and we get the roots of (2.1).

Now let us proceed with the solution of (2.2). If we choose the change of variable $z=x-(a/4)$, the equation (2.2) changes to the form

$$x^4 + Ax^2 + Bx + C = 0. \tag{2.4}$$

We choose $y=x^2$. Then getting the roots of (2.4) is the same as solving the system of equations

$$\begin{cases} x^2 = y \\ y^2 + Ay + Bx + C = 0 \end{cases} \tag{2.5}$$

for x.

It is easily seen that the equations of (2.5) are the equations of two parabolas whose axes are perpendicular to one another. The solution of (2.5) is obtained by the system

$$\begin{cases} x^2 = y \\ x^2 + y^2 + (A - 1)y + Bx + C = 0. \end{cases} \tag{2.6}$$

The advantage of (2.6) is that the parabola $y=x^2$ can be drawn accurately on a sheet of scaled paper. Then a circle of center

$$\left(\frac{A-1}{2}, \frac{B}{2}\right)$$

and radius

$$\frac{\sqrt{A^2+B^2-4AC-2A+1}}{2}$$

can be drawn on a sheet of transparent paper. We superpose the circle on the parabola and read the roots on the x-axis (Fig. 1).

3. An example: The equation

$$x^3+Bx=C \tag{3.1}$$

is solved by Khayyam in the following way. In this equation B and C are supposed positive. We choose b and c such that

$$b^2=B \quad \text{and} \quad b^2c=C.$$

Then (3.1) is written as

$$x^3+b^2x=b^2c.$$

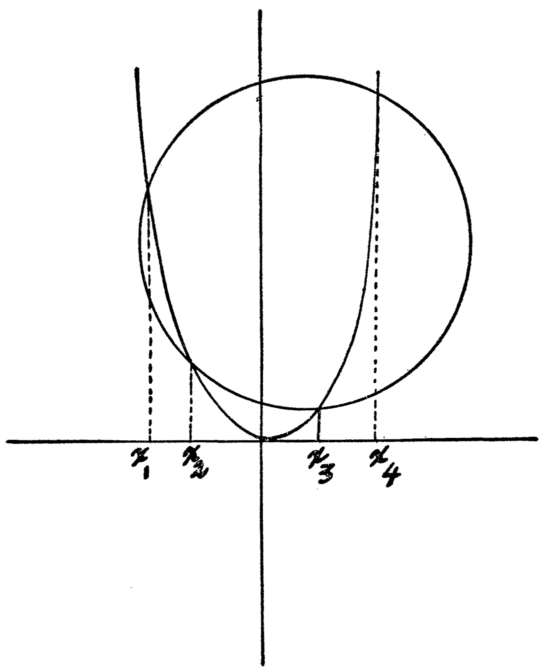

FIG. 1

Choose the segment $AB=b$ and $BC=c$ (Fig. 2). AB is perpendicular to BC.

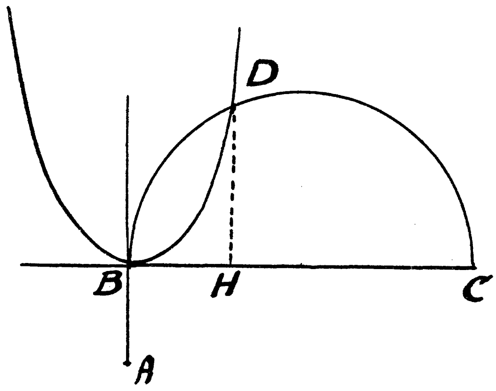

FIG. 2

We draw the half circle with diameter BC, and the parabola with (normal side) AB. This means the parabola $x^2=by$. These curves intersect at D. We draw the perpendicular DH to BC. Then

$$x = DH.$$

Omar proved synthetically that DH is a root of (3.1). The proof is:

$$BH^2 = (AB)(DH).$$

Thus

$$\frac{AB}{BH} = \frac{BH}{DH} \quad \text{or} \quad \frac{b}{x} = \frac{x}{DH}. \tag{3.2}$$

But in the circle

$$\frac{BH}{HD} = \frac{HD}{HC}.$$

Thus

$$\frac{AB}{BH} = \frac{DH}{HC}, \quad \text{or} \quad \frac{b}{x} = \frac{DH}{c-x}. \tag{3.3}$$

Finding DH from (3.2) and (3.3) and comparing them we get

$$x^3 + b^2x = b^2c.$$

Thus the equation has one real root. This root always exists, and is obtained by the intersection of a circle and a parabola.

To explain Omar Khayyam's method we used analytic geometry. Actually in Omar's work every problem was done synthetically.

ON THE GEOMETRY OF THE n-DIMENSIONAL SIMPLEX

SHIKŌ IWATA, Gifu University, Japan

1. Introduction. We consider an n-dimensional simplex S_{n+1} in the Euclidean space E_n and denote its vertices by A_i $(i=1, 2, \cdots, n+1)$. The components of $\mathfrak{a}_i$ which is the vector of A_i will be denoted by $a_{i\alpha}$ $(\alpha=1, 2, \cdots, n)$ referred to a rectangular co-ordinate system. Then the volume V of the simplex S_{n+1} will be expressed in the form

$$(1)\qquad n!V = \begin{vmatrix} 1 & a_{11} & a_{12} & \cdots & a_{1n} \\ 1 & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} = \sum_{i=1}^{n+1} (-1)^{i-1}[\mathfrak{a}_i],$$

where $[\mathfrak{a}_i]$ is the determinant excluding the i-th row and the 1st column of the above determinant.

If we denote by V_i the volume of the simplex obtained by replacing vertex A_i by the origin O, we shall have the relation

$$V = \sum V_i,$$

where $\sum$ means $\sum_{i=1}^{n+1}$. More generally if V_i^p denotes the volume of the simplex replacing the vertex A_i by any point P, we have

$$(2)\qquad V = \sum V_i^p.$$

From (1) we have the relation

$$\sum(-1)^{i-1}[\mathfrak{a}_i]\mathfrak{a}_i = \sum (n!V_i)\mathfrak{a}_i = 0$$

more generally

$$(3)\qquad \sum n!V_i^p\mathfrak{a}_i = (n!V)\mathfrak{p}$$

where $\mathfrak{p}$ denotes the vector of the point P.

If we consider a vector $\mathfrak{a}^i$ replacing the i-th row of the determinant (1) by the axial unit vectors $\mathfrak{e}_\alpha$, i.e.,

$$\mathfrak{a}^i = \begin{vmatrix} 1 & a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots & \mathfrak{e}_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \qquad (i\text{-th row})$$

we have the following relations,

$$(4)\qquad \|\mathfrak{a}^i\| = (n-1)!\|M_i\|,$$

where $\|M_i\|$ represents the absolute value of the volume of the $n-1$ dimensional simplex constructed by the vertices excluding A_i of the simplex S_{n+1}. We call the vector $\mathfrak{a}^i$ the Areal Vector. The direction of $\mathfrak{a}^i$ is orthogonal to the hyperplane constructed by the vertices other than A_i of the simplex S_{n+1}.

$$\sum \mathfrak{a}^i = 0 \tag{5}$$

$$\overset{i}{\mathfrak{a}}\mathfrak{p} + n!V_i = n!V_i^{\mathfrak{p}}, \tag{6}$$

where the product of vectors means scalar product.

$$\sum (\mathfrak{a}^i\mathfrak{p})\mathfrak{a}_i = (n!V)\mathfrak{p}. \tag{7}$$

The vector equation of the hyperplane passing through the n vertices of the simplex S_{n+1} excluding A_i, is

$$\mathfrak{a}^i\mathfrak{r} + n!V_i = 0, \tag{8}$$

where $\mathfrak{r}$ denotes the current vector. We call this hyperplane the face of S_{n+1} corresponding to A_i. The vector equation of the circum-hypersphere of the simplex S_{n+1} is written

$$(n!V)\mathfrak{r}^2 - \sum \mathfrak{a}_i^2\overset{i}{\mathfrak{a}}\mathfrak{r} - \sum \mathfrak{a}_i^2(n!V_i) = 0. \tag{9}$$

Generally the equation of the hypersphere cutting orthogonally the $n+1$ hyperspheres having A_i as its center and R_i as its radius, is

$$(n!V)\mathfrak{r}^2 - \sum (\mathfrak{a}_i^2 - R_i^2)\overset{i}{\mathfrak{a}}\mathfrak{r} - \sum (\mathfrak{a}_i^2 - R_i^2)(n!V_i) = 0. \tag{10}$$

2. THEOREMS. From the preceding formulas we have the following theorems.

Theorem 1. *Let A_i' be the intersection of a line $\mathfrak{l}$ and the face corresponding to A_i of the simplex S_{n+1}, then the $n+1$ hyperspheres having A_iA_i' as diameters pass through two fixed points.*

Proof. The equation of the hypersphere having A_iA_i' as its diameter is

$$(\mathfrak{r} - \mathfrak{a}_i)(\mathfrak{r} - \mathfrak{a}_i') = 0.$$

Since A_i' $(\mathfrak{a}_i')$ is the intersection of the hyperplane (8) and the line $\mathfrak{r}=t\mathfrak{l}$, we have

$$\mathfrak{a}_i' = -\frac{n!V_i}{\mathfrak{a}^i\mathfrak{l}}\mathfrak{l}.$$

Therefore the preceding equation may be written in the form

$$\mathfrak{r}^2 - \left(\mathfrak{a}_i - \frac{n!V_i}{\mathfrak{a}^i\mathfrak{l}}\mathfrak{l}\right)\mathfrak{r} - \frac{n!V_i}{\mathfrak{a}^i\mathfrak{l}}(\mathfrak{a}_i\mathfrak{l}) = 0,$$

or

$$(\mathfrak{a}^i\mathfrak{l})\mathfrak{r}^2 - \{(\mathfrak{a}^i\mathfrak{l})\mathfrak{a}_i - (n!V_i)\mathfrak{l}\}\mathfrak{r} - (n!V_i)(\mathfrak{a}_i\mathfrak{l}) = 0.$$

The foregoing $n+1$ equations for $i=1, 2, \cdots, n+1$ are linearly dependent, because if we add them side by side we have an identically vanishing equation, as can be seen from the formulas (1, 5), (1, 7), (1, 2), (1, 3). Hence the theorem is proved.

COROLLARY 1. *If the line* $\mathfrak{l}$ *passes through a point* P *whose barycentric co-ordinates are* α_i $(i=1, 2, \cdots, n+1)$, *we have the relation*

$$\sum \frac{\alpha_i}{PA_i'} = 0.$$

COROLLARY 2. *If we translate the vectors* A_iA_i' *to the fixed point, the volume of the simplex constructed by the end points of these vectors will be* $2V$.

THEOREM 2. *Let* V, V' *be the volumes of the simplex* S_{n+1}', S_{n+1}, *then we have the relation*

$$\sum_i V_i^{k'} V_{k'}^{\prime i} = VV' \qquad (K = 1, 2, \cdots, n+1),$$

where $V_i^{k'}$ *denotes the volume of the simplex replacing the vertex* A_i *of* S_{n+1} *by the vertex* A_k' *of* S_{n+1}'.

Proof. Using (7), we have

$$\sum (\mathfrak{a}^i \mathfrak{a}_k')\mathfrak{a}_i = (n!V)\mathfrak{a}_k',$$

therefore

$$\sum_i (\mathfrak{a}^i \mathfrak{a}_k')(\mathfrak{a}_i \mathfrak{a}'^k) = (n!V)(\mathfrak{a}_k' \mathfrak{a}'^k)$$

Adding the identity

$$\sum (n!V_i)\mathfrak{a}_i \mathfrak{a}'^k = 0,$$

the above relation becomes

$$(\mathfrak{a}^i \mathfrak{a}_k' + n!V_i)(\mathfrak{a}_i \mathfrak{a}'^k) = (n!V)(\mathfrak{a}_k' \mathfrak{a}'^k).$$

If we choose any other vertex of A_k' of S_{n+1}' as the origin, we shall have the relations

$$\overset{i}{\mathfrak{a}}\overset{\prime}{\mathfrak{a}_k} + n!V_i = n!V_i^{k'}, \qquad \mathfrak{a}_i \mathfrak{a}'^k = n!V_{k'}^{\prime i}, \qquad \mathfrak{a}_k' \mathfrak{a}'^k = n!V'.$$

By using these relations we can prove the theorem at once.

THEOREM 3. *If the length of the common perpendicular to the two flat spaces one of which is passing through the* p *vertices of the simplex* S_{n+1} *and the other of which is passing through the other vertices be denoted by* $d_{i_1 i_2 \ldots i_p}$ *and the length of the perpendicular dropped from* A_i *upon the corresponding face of* S_{n+1} *be denoted by* h_i, *then we have the relation*

$$\sum_i \frac{1}{(d_{i_1 i_2} \cdots {}_{i_p})^2} = {}_{n-1}C_{p-1} \sum \frac{1}{h_i^2} \qquad (p > 1, \text{ fixed})$$

where $\sum_i$ indicates summation on i.

Proof. If we construct a hyperplane which is parallel to the flat space passing through the vertices $A_{i_1}, A_{i_2}, \cdots, A_{i_p}$ of the simplex S_{n+1} and passes through the other $n+1-p$ vertices, then $d_{i_1,i_2 \ldots i_p}$ is equal to the length of the perpendicular dropped from any point of $A_{i_1}, A_{i_2}, \cdots, A_{i_p}$ upon that hyperplane, and the equation of this hyperplane may be written in the form

$$(\mathfrak{a}^{i_1}\mathfrak{r} + n!V_{i_1}) + (\mathfrak{a}^{i_2}\mathfrak{r} + n!V_{i_2}) + \cdots + (\mathfrak{a}^{i_p}\mathfrak{r} + n!V_{i_p}) = 0,$$

because this hyperplane passes through the other vertices of $A_{i_1}, A_{i_2}, \cdots, A_{i_p}$ and the vector

$$\mathfrak{a}^{i_1} + \mathfrak{a}^{i_2} + \cdots + \mathfrak{a}^{i_p} = -(\mathfrak{a}^{i_{p+1}} + \mathfrak{a}^{i_{p+2}} + \cdots + \mathfrak{a}^{i_{n+1-p}})$$

is orthogonal to the flat space passing through $A_{i_1}, A_{i_2}, \cdots, A_{i_p}$.

Therefore

$$d_{i_1 i_2 \cdots i_p} = \frac{\left\|(\mathfrak{a}^{i_1}\mathfrak{a}_i + n!V_{i_1}) + (\mathfrak{a}^{i_2}\mathfrak{a}_{i_1} + n!V_{i_2}) + \cdots + (\mathfrak{a}^{i_p}\mathfrak{a}_{i_1} + n!V_{i_p})\right\|}{\left\|\mathfrak{a}^{i_1} + \mathfrak{a}^{i_2} + \cdots + \mathfrak{a}^{i_p}\right\|}$$

$$= \frac{n!\|V\|}{\left\|\mathfrak{a}^{i_1} + \mathfrak{a}^{i_2} + \cdots + \mathfrak{a}^{i_p}\right\|},$$

or

$$(n!V)^2 \sum_i \frac{1}{(d_{i_1 i_2 \cdots i_p})^2} = \sum_i (\mathfrak{a}^{i_1} + \mathfrak{a}^{i_2} + \cdots + \mathfrak{a}^{i_p})^2$$

$$= \sum_i \left\{(\mathfrak{a}^{i_1})^2 + (\mathfrak{a}^{i_2})^2 + \cdots + (\mathfrak{a}^{i_p})^2\right\} + 2\sum_i \sum_{\alpha \neq \beta} \mathfrak{a}^{i_\alpha}\mathfrak{a}^{i_\beta}$$

$$= {}_nC_{p-1} \sum (\mathfrak{a}^i)^2 + 2\,{}_{n-1}C_{p-2} \sum_{i \neq j} \mathfrak{a}^i\mathfrak{a}^j.$$

By (1, 5) we have

$$2\sum_{i \neq j} \mathfrak{a}^i\mathfrak{a}^j = -\sum (\mathfrak{a}^i)^2.$$

Therefore the right hand side of the above relation becomes

$$({}_nC_{p-1} - {}_{n-1}C_{p-2}) \sum (\mathfrak{a}^i)^2 = {}_{n-1}C_{p-1} \sum (\mathfrak{a}^i)^2.$$

By using the relation $\|\mathfrak{a}^i\| h_i = n!\|V\|$, we can prove the theorem.

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SIMILAR SOLUTIONS TO THE GENERALIZED PLANAR THREE BODY PROBLEM

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A. Introduction. We shall assume that the force on a particle with mass m_i due to another with mass m_j is

$$-Gm_im_j\frac{(r_i-r_j)}{|r_i-r_j|^{n+1}}$$

where n is a real number greater than zero; G, a positive constant; and r_k the position vector of p_k, the kth particle with mass m_k. For $n=2$ this represents the gravitational attraction between two bodies.

In this paper our attention shall be focused on all solutions to the planar three body problem that are similar, i.e., have the property that at each instant of time the sides of the triangle (possibly degenerate) formed by the particles are proportional to the corresponding sides of the initial configuration.

We shall show that:

(i) The instantaneous triangle is always (a) equilateral (but not necessarily congruent to the initial configuration) or (b) degenerate.

(ii) These solutions are easily obtained from the solutions to the two body problem for the same force field.

B. Necessary and Sufficient Conditions for Similar Solutions. Let Γ be the inertial coordinate system with the origin at the center of mass and Γ' the coordinate system with the origin on p_1 and axes parallel to those of Γ. Let (x_k, y_k) be the coordinates of p_k, and w_k and z_k the representations of the position of p_k by means of complex numbers in Γ and Γ' respectively, i.e.,

$$w_k = x_k + iy_k$$

and

$$z_k = w_k - w_1.$$

The equations of motion in Γ are:

$$m_1\frac{d^2w_1}{dt^2} = -Gm_1m_2\frac{(w_1-w_2)}{|w_1-w_2|^{n+1}} - Gm_1m_3\frac{(w_1-w_3)}{|w_1-w_3|^{n+1}}$$

$$m_2\frac{d^2w_2}{dt^2} = -Gm_2m_1\frac{(w_2-w_1)}{|w_2-w_1|^{n+1}} - Gm_2m_3\frac{(w_2-w_3)}{|w_2-w_3|^{n+1}}$$

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$$m_3 \frac{d^2 w_3}{dt^2} = - G m_3 m_1 \frac{(w_3 - w_1)}{|w_3 - w_1|^{n+1}} - G m_3 m_2 \frac{(w_3 - w_2)}{|w_3 - w_2|^{n+1}}$$

from which the equations of motion in Γ' can be found to be:

$$(1)\qquad \begin{aligned} \frac{d^2 z_2}{dt^2} &= - G(m_1 + m_2) \frac{z_2}{|z_2|^{n+1}} - G m_3 \left[\frac{z_2 - z_3}{|z_2 - z_3|^{n+1}} + \frac{z_3}{|z_3|^{n+1}} \right] \\ \frac{d^2 z_3}{dt^2} &= - G(m_1 + m_3) \frac{z_3}{|z_3|^{n+1}} - G m_2 \left[\frac{z_3 - z_2}{|z_3 - z_2|^{n+1}} + \frac{z_2}{|z_2|^{n+1}} \right]. \end{aligned}$$

Using the cosine law for triangles it is easy to verify that the similarity condition is equivalent to the condition that the ratio $|z_2|/|z_3|$ and α, the angle of rotation from z_3 to z_2 remain constant. Thus

$$(2)\qquad z_2 = c z_3$$

and

$$(3)\qquad z_3 - z_2 = (1 - c) z_3 \quad \text{where} \quad c = \frac{|z_2|}{|z_3|} e^{i\alpha}.$$

We shall assume that $c \neq 0$ and $c \neq 1$ since either of these implies that two of the bodies remain together.

Substituting (2) and (3) into (1) we obtain:

$$(4)\qquad \begin{aligned} \frac{d^2 z_3}{dt^2} + G S_2 \frac{z_3}{|z_3|^{n+1}} &= 0 \\ \frac{d^2 z_3}{dt^2} + G S_3 \frac{z_3}{|z_3|^{n+1}} &= 0 \end{aligned}$$

where

$$(5)\qquad \begin{aligned} S_2 &= \frac{1}{c} \left[(m_1 + m_2) \frac{c}{|c|^{n+1}} + m_3 \left(1 - \frac{(1 - c)}{|1 - c|^{n+1}} \right) \right] \\ S_3 &= \left[(m_1 + m_3) + m_2 \left(\frac{(1 - c)}{|1 - c|^{n+1}} + \frac{c}{|c|^{n+1}} \right) \right]. \end{aligned}$$

If $S_2 \neq S_3$ (4) is inconsistent since for any $z_3 \neq 0$ it would give two different values for $d^2 z_3/dt^2$. If $S_2 = S_3$ and $z_3(t)$ is a solution to (4), $z_2 = c z_3(t)$ and $z_3(t)$ are solutions to (1). Therefore a necessary and sufficient condition that a solution is similar is that $c(S_2 - S_3) = 0$ or

$$(6)\qquad g(m_1, m_2, m_3, c, 1 - c) = 0$$

where

$$g(m_1, m_2, m_3, c, 1 - c) = m_1 c \left(\frac{1}{|c|^{n+1}} - 1 \right) + m_2 c (1 - c) \left(\frac{1}{|c|^{n+1}} - \frac{1}{|1 - c|^{n+1}} \right)$$

$$+ m_3(1 - c)\left(1 - \frac{1}{|1 - c|^{n+1}}\right).$$

C. Equilateral Triangle Case. Let us consider the solutions of (6) that do not depend on the masses, i.e., $|c| = |1-c| = 1$. In this case $c = e^{(\pi/3)i}$ or $c = e^{-(\pi/3)i}$ and the instantaneous triangle is always equilateral.

D. Collinear Case. Let us now consider the real solutions to equation (6). If a similar solution exists for a real c, the three bodies lie on a straight line. Given any three particles p_α, p_β, p_γ with masses m_α, m_β and m_γ respectively, let us label them p_1, p_2 and p_3 arbitrarily and look for a solution such that p_2 lies on the line segment joining the other two, i.e.,

$$0 < c < 1. \tag{7}$$

We shall see that under this assumption equation (6) has only one real root for a given m_1, m_2 and m_3. Although we can select p_1, p_2 and p_3 from p_α, p_β, and p_γ in six different ways, there are at most 3 different configurations if c is real since:

$$g(m_1, m_2, m_3, c, 1 - c) = - g(m_3, m_2, m_1, 1 - c, c).$$

From (6) and (7) we obtain:

$$h(c) = 0$$

where

$$h(c) = \left[\frac{m_1}{c^n} + m_2\frac{(1 - c)}{c^n} + m_3(1 - c)\right] - \left[m_1 c + m_2\frac{c}{(1 - c)^n} + \frac{m_3}{(1 - c)^n}\right]$$

$h(c)$ is a strictly decreasing function of c in the interval of interest since the first bracket is a strictly decreasing function while the second is a strictly increasing function. Furthermore,

$$\lim_{c\to 0} h(c) = +\infty \qquad \lim_{c\to 1} h(c) = -\infty.$$

Therefore, $h(c) = 0$ for only one value of c between zero and one.

E. Other Possible Solutions. Now let $c = a + ib$. As above if $b = 0$ we assume that $0 < c < 1$. Using (6) it is easy to show that for $|c| = 1$ and $|1-c| \neq 1$, $cm_2 + m_3 = 0$, while for $|c| \neq 1$ and $|1-c| = 1$, $m_1 + (1-c)m_2 = 0$. The first implies that $c < 0$ and the second that $c > 1$. Therefore, the only remaining cases to consider are those for which all of the following are true:

$$b \neq 0, \qquad |c| \neq 1 \quad \text{and} \quad |1 - c| \neq 1.$$

By interchanging z_2 and z_3 if necessary we may assume:

$$|c| < 1. \tag{8}$$

Substituting $c = a + ib$ into (6) we obtain:

$$\begin{aligned} aM_1 + (1 - a)M_3 &= (a - a^2 + b^2)M_2 \\ M_1 - M_3 &= (1 - 2a)M_2 \end{aligned} \tag{9}$$

where

$$M_1 = m_1\left(\frac{1}{|c|^{n+1}} - 1\right)$$

$$M_2 = m_2\left(\frac{1}{|1-c|^{n+1}} - \frac{1}{|c|^{n+1}}\right)$$

$$M_3 = m_3\left(1 - \frac{1}{|1-c|^{n+1}}\right).$$

The solution of (9) is

$$M_3 = M_2|c|^2$$

$$M_1 = M_2|1-c|^2.$$

This implies (we assume $M_2 \neq 0$, since $M_2=0$ implies $n_1=n_3=0$)

$$\text{(10)} \qquad \operatorname{sgn} M_1 = \operatorname{sgn} M_3 = \operatorname{sgn} M_2 \quad \text{where} \quad \operatorname{sgn} x = 1 \text{ if } x > 0 \text{ and } \operatorname{sgn} x = -1 \text{ if } x < 0.$$

Because of (8)

$$\text{(11)} \qquad \operatorname{sgn} M_1 = \operatorname{sgn} m_1.$$

If $|1-c| < 1$ the definition of M_3 implies sgn $M_3 = -$sgn m_3. This with (10) and (11) implies that either m_1 or m_3 is less than zero. Similarily if $|1-c| > 1$, sgn $M_2 = -$sgn m_2 and either m_1 or m_2 is less than zero.

Therefore, the only solutions are those discussed in C and D.

F. Connection With the Two Body Problem. Let us assume, as above, that either $0 < c_0 < 1$ and $h(c_0) = 0$ or $|c| = |1-c| = 1$. In both cases from (5) we obtain $S_3 > 0$ and the differential equations (1) are equivalent to:

$$\frac{d^2 z_3}{dt^2} + GS\frac{z_3}{|z_3|^{n+1}} = 0$$

$$z_2 = cz_3$$

$$S > 0 \quad \text{where} \quad S = S_2 = S_3.$$

However, the equation for z_3 is the equation of motion for a particle relative to another if no other bodies are present and if the sum of their masses is S. Thus in Γ' p_1 remains at the origin while the trajectory of p_3 is what it would be if m_2 were zero and m_1+m_3 were equal to S. The trajectory of p_2 is similar to that of p_3, but is scaled by a factor c_0 (collinear case) or is the same as that of p_3 except that it is rotated through an angle of $\pi/3$ radians (equilateral triangle case).

In Γ the situation is a little more complicated. Since

$$\sum_{j=1}^{3} m_j w_j = 0,$$

it follows that

$$w_k = z_k + w_1 = z_k - \frac{m_2 c + m_3}{m_1 + m_2 + m_3} z_3$$

or

$$w_k = l_k z_3$$

where l_k is a complex constant with the polar form $c_k e^{i\alpha_k}$. Thus w_1, w_2 and w_3 describe trajectories that are similar to that traced by z_3, but scaled by the factor c_k and rotated by the angle α_k.

In the classical case $n=2$, the orbit of p_3 lies on a degenerate or non-degenerate conic. It is interesting to note that for $n=2$ we can construct both collinear and equilateral solutions that involve a triple collision. Also for both the collinear and equilateral triangle case the eccentricity of the orbits of all particles in Γ is the same since multiplication by a complex constant l_k does not change the eccentricity.

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ON A CONJECTURE OF MURPHY, II

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In [3] R. Murphy stated the following conjecture:

If a is a primitive root of a prime number p and $(a/2) < p < a$, then a is a primitive root of every prime number an^2+p.

In [2] I disproved this conjecture, but its special case for $a=10$, $p=7$, which led Murphy to his general statement remained, however, not settled.

From the tables given in the chapter V of [1] it follows that in this case the conjecture is false, too. The numbers $q_n = 10n^2+7$ $(n=27, 30, 36)$ are prime and provide the counter-examples

$$10^{1/3(q_n-1)} \equiv 1 \pmod{q_n}.$$

The other numbers q_n $(n<38)$ satisfy the conjecture of Murphy.

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SEMI-COMPLEX FUNCTIONS AND THEIR GRAPHS

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In real variable theory there are many common functions where the range of the dependent variable is limited. For example, in $y=\sin x$, $|y| \leqq 1$, for all real x; and in $y=e^x$, $y>0$ for all real x. If we now consider y as the independent variable, and assign to it real values outside of its range corresponding to real x, we can expect that x will, in general, be a complex number. It is the purpose of this paper to analyze some specific functional relations between x and y under such conditions; and to observe the graphs of such functions as curves in a three-dimensional space, two dimensions of which are associated with complex x and the third with real y.

As an introductory example, consider the quadratic $y=x^2-6x+25$. It is easily verified that the minimum point is (3, 16) and hence the range of y, corresponding to real x, is $y \geqq 16$. Let us now assign to y the value 15. We find that the corresponding values of x are $x=3\pm i$, and in general for any $y<16$, x will be a pair of conjugate complex numbers.

We then define a semi-complex functional relation between x and y as one where to any real y there correspond values of x, which may be either real or complex. A semi-complex functional relation may be represented graphically by a three-dimensional rectangular coordinate system consisting of a complex x plane and a real y axis. Let $x \equiv u+iv$. Hence, a point in this space will have coordinates (u, v, y). For example, the functional values cited above, $y=15$, $x=3\pm i$, would represent the points (3, 1, 15) and (3, −1, 15). For values of y corresponding to real x, v would be zero and such points would lie in the $v=0$ plane. Thus when $y=20$, x has real values 1 and 5, and we have the points (1, 0, 20) and (5, 0, 20). By this procedure it is possible to investigate the nature of the graphs of the equations of semi-complex functions over an unlimited range of real y.

Let us continue with the example $y=x^2-6x+25$. The following table of values is easily verified:

$x=u+iv$	y
0 and 6	25
1 and 5	20
2 and 4	17
3	16
$3\pm i$	15
$3\pm 2i$	12
$3\pm 3i$	7
$3\pm 4i$	0
$3\pm 5i$	−9

These functional values are plotted as points in (u, v, y) space and connected by a smooth curve as in Fig. 1.

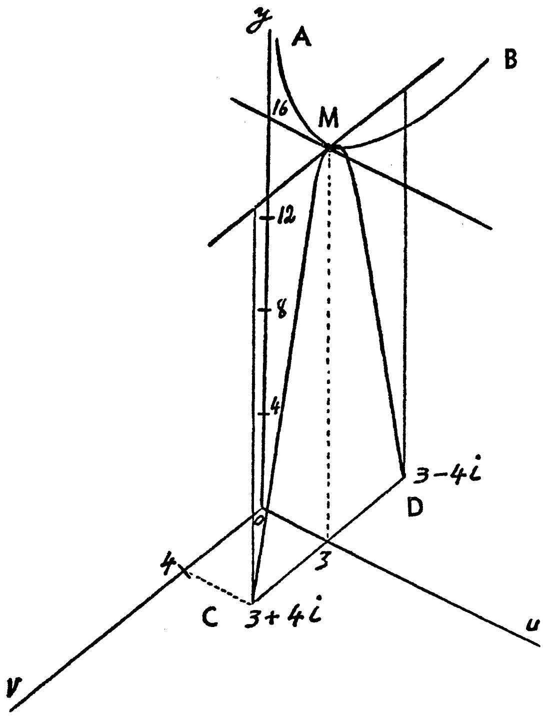

FIG. 1

That portion of the curve AMB for $y \geqq 16$ has real values of x and is a parabola in the $v=0$ plane. This, of course, is the conventional curve obtained by plotting $y=x^2-6x+25$ by the usual methods of analytic geometry. However, we now see that for real values of $y<16$ another curve CMD emerges with complex values for $x \equiv u+iv$. This curve lies in the plane $u=3$, touches the real curve at point M, (3, 0, 16), is concave downward, and pierces the x plane at $3 \pm 4i$. It will be shown later that this curve is congruent to the real curve. The traces in the x plane, $3 \pm 4i$, are the complex roots of $x^2-6x+25=0$, and thus we have a geometric representation of the complex roots of a quadratic equation.

Let us now investigate $y=\sin x$, considering it as a semi-complex function. As above, let $x \equiv u+iv$, then $y=\sin(u+iv)$. From the theory of complex variables we list for reference the following well-known identities [1].

(1) $$\sin(A + iB) \equiv \sin A \cosh B + i \cos A \sinh B$$

(2) $$\cos(A + iB) \equiv \cos A \cosh B - i \sin A \sinh B$$

(3) $$\cosh(A + iB) \equiv \cosh A \cos B + i \sinh A \sin B$$

(4) $$\sinh(A + iB) \equiv \sinh A \cos B + i \cosh A \sin B$$

Applying identity (1) to the case under consideration, we have

$$y = \sin(u + iv) = \sin u \cosh v + i \cos u \sinh v.$$

Since in a semi-complex function y is always real, its imaginary part must vanish.

Hence,

$$\cos u \sinh v = 0. \tag{5}$$

This implies that $\cos u = 0$, or $\sinh v = 0$. Consider the latter condition as case A. If $\sinh v = 0$, then $v = 0$, and $y = \sin(u + iv)$ reduces to $y = \sin u$. The graph, Fig. 2 is, of course, the familiar sine curve lying in the $v = 0$ plane. As case B, consider $\cos u = 0$. The leads to $u = \pi/2 + \pi n$, where n is any integer. Now $y = \sin(u + iv)$ becomes,

$$y = \sin\left(\frac{\pi}{2} + n\pi\right)\cosh v, \quad \text{or} \tag{6}$$

$$y = (-1)^n \cosh v.$$

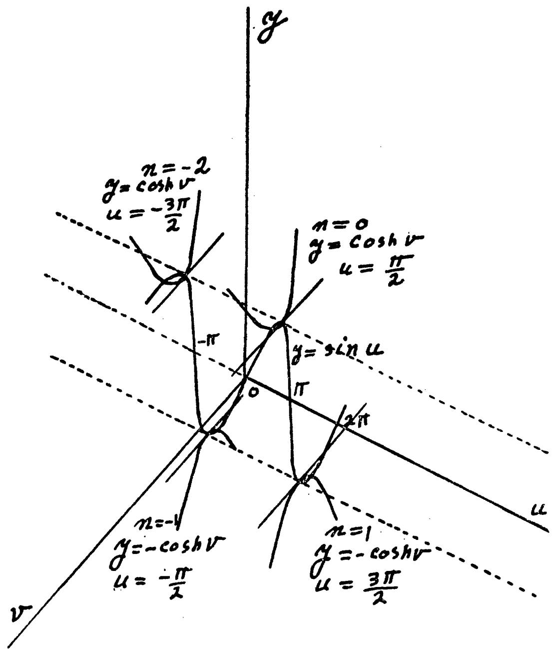

FIG. 2

The geometric interpretation of this analysis is shown in Fig. 2. In each of the set of planes $u = \pi/2 + n\pi$ lies a hyperbolic cosine curve given by equation (6). These hyperbolic cosines are, of course, oriented at right angles to the sine curve of case A. The extreme points of the hyperbolic cosine curves coincide with the extreme points of the sine curve. There would seem to be an opportunity here for the mathematical craftsman to create a graceful and interesting mobile.

As might be intuitively surmised, an analysis of $y = \cos x$ from the semi-complex viewpoint, produces a graphical configuration similar to Fig. 2, except that there is now a cosine curve in the $v = 0$ plane. Hyperbolic cosine curves extend off the extreme points of the cosine curve and are oriented at right angles to it, lying in the planes $u = n\pi$, for integral n. This is shown in Fig. 3.

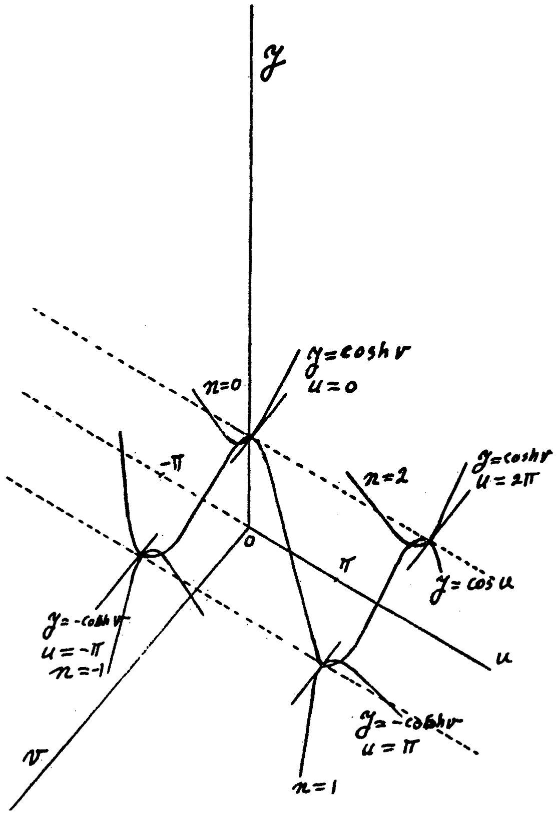

FIG. 3

Consider now the relation $y=e^x$ as a semi-complex function. Since $x\equiv u+iv$, $y=e^{u+iv}=e^u\cos v+ie^u\sin v$. Since y is real,

$$e^u \sin v = 0 \tag{7}$$

Hence,

$$v = n\pi, \qquad \text{for integral } n, \tag{8}$$

and

$$y = e^u \cos n\pi, \quad \text{or}$$

$$y = (-1)^n e^u. \tag{9}$$

The graphical interpretation of this analysis is shown in Fig. 4, where we see a set of exponential curves of the form $y=(-1)^n e^u$, each lying in one of the set of planes $v=n\pi$. The usual exponential curve from plane analytic geometry is a special case of this system for $n=0$. When n is odd, we have a set of curves $y=-e^u$, corresponding to negative values of y.

An unexpected result of the analysis indicates that for n even, there is a multiplicity of x values for $y>0$. Thus we find an infinite set of curves of the form $y=e^u$, each lying in a plane $v=n\pi$, (n even). For example, when $n=2$, we

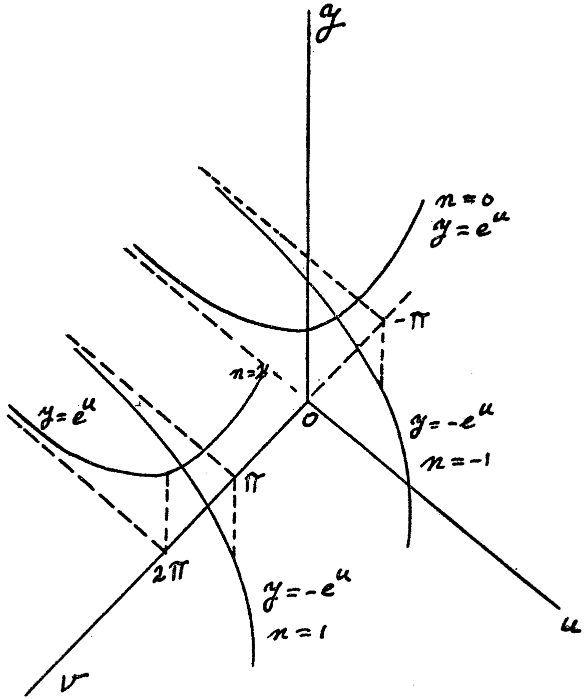

FIG. 4

have the curve $y=e^u$ in the plane $v=2\pi$, shown as the foremost curve in Fig. 4.

The function $y=\cosh x$ will now be analyzed as a semi-complex function, by means of identity (3). As usual, $x\equiv u+iv$, hence

$$y = \cosh(u+iv), \quad \text{or} \qquad (10)$$
$$y = \cosh u \cos v + i \sinh u \sin v.$$

Since y is real, $\sinh u \sin v=0$, and again, as with the analysis of $\sin x$, there are two cases:

Case A: Sinh $u=0$. This implies $u=0$ and hence equation (10) reduces to

$$y = \cos v. \qquad (11)$$

Geometrically, we have a cosine curve, $y=\cos v$, lying in the plane $u=0$, as shown in Fig. 5.

Case B: Sin $v=0$. This implies $v=n\pi$, for integral n. Hence,

$$y = \cos n\pi \cosh u, \quad \text{or} \qquad (12)$$
$$y = (-1)^n \cosh u.$$

This is represented in Fig. 5 by a set of hyperbolic cosine curves, lying in planes $v=n\pi$, oriented at right angles to the cosine curve and touching it at the extreme points. The common hyperbolic cosine curve of the real theory is the case when $n=0$.

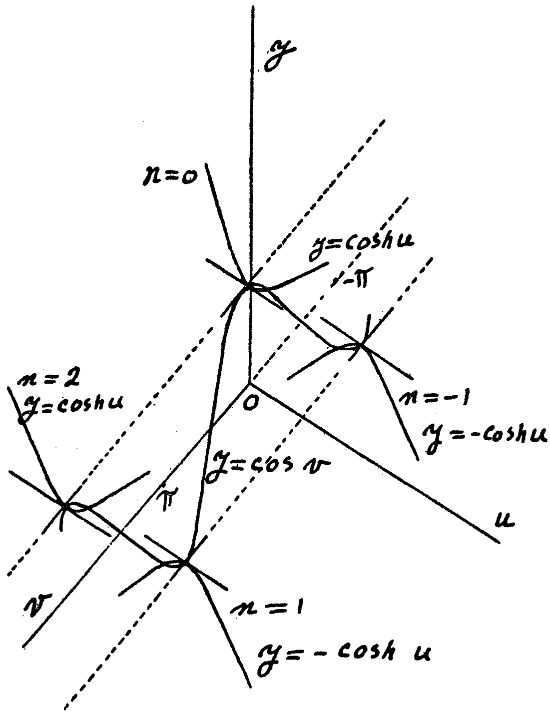

FIG. 5

There are two interesting aspects of this geometry. First, we note that for the range $y>1$, the functional relation $y=\cosh x$ is satisfied by complex values of x in addition to the usual real values. This is shown in the graph by hyperbolic cosines for even n, which extend upward from the maxima of the cosine curve. Secondly, the graphical nature of the functional relation between y and x for $y \leqq 1$ consists of two dissimilar types of curves, namely the cosine curve for $|y| \leqq 1$ and the infinite set of hyperbolic cosine curves for $y<-1$.

The obvious similarity between the graph of $y=\cos x$, Fig. 2; and $y=\cosh x$, Fig. 5, is explained by the identity from complex variable theory

$$\cosh A \equiv \cos iA.$$

Thus $y=\cosh x$ may be written $y=\cos ix$, and we see that the graph of $y=\cosh x$ may be obtained by a 90° rotation of the graph of $y=\cos x$ about the y axis due to the i coefficient of x.

Now let us return to a more careful consideration of the quadratic $y=x^2-6x+25$ which served as an introductory example. Following the usual procedure, we let $x \equiv u+iv$ and separate the real and imaginary terms to obtain

$$y = (u^2 - v^2 - 6u + 25) + (2uv - 6v)i. \tag{13}$$

Since y is real, $2uv-6v=0$. Hence, $u=3$ or $v=0$. Referring to Fig. 1, we see that these are the planes in which the two branches of the graph lie. Considering first $v=0$, we find that equation (13) reduces to

$$y = u^2 - 6u + 25. \tag{14}$$

This is, of course, the usual parabola from real theory. Next, consider $u=3$, then again from (13), we have

$$y = -v^2 + 16. \tag{15}$$

This is the equation of the parabola in Fig. 1 which opens downward and has complex x values. Since the shape of the parabola is completely determined by the coefficient of the second degree term, the two branches are congruent as originally asserted.

A function may have real values of x for the complete range of real y values and still possess distinct branches with complex x values. Such a function is $y=\sinh x$, which in real theory has an unlimited y range. If we analyze this function from the semi-complex viewpoint, using (4) we find a set of curves $y=(-1)^n \sinh u$, each of which lies in a plane of the set $v=n\pi$ (n integral). The curve associated with $n=0$ is the curve from the real theory, but we see that, in general, for any real y, there is a multiplicity of complex x's which satisfy the functional relation $y=\sinh x$.

As a final example, consider the function $y=x^3$. Again in the real theory y is unlimited. Following the established procedure, we let $x\equiv u+iv$, and then

$$y = (u^3 - 3uv^2) + v(3u^2 - v^2)i. \tag{16}$$

If y is to be real, we must have $v=0$ or $v=\pm\sqrt{3}u$. The first condition leads to $y=u^3$, which is the usual cubic curve in the plane $v=0$ from real theory. However, if we use $v=\pm\sqrt{3}u$, we have $y=-8u^3$. The geometric interpretation of this latter case is two space curves formed by the intersection of the cylinder $y=-8u^3$ and the planes $v=\pm\sqrt{3}u$ respectively. Again we see that for any real y, x is in general complex.

Reference

1. C. R. Wylie, Jr., Advanced engineering mathematics, sec. ed., New York (1960) p. 554–555.

TEACHING OF MATHEMATICS

EDITED BY ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.

A COMMON MISAPPLICATION OF THE AXIOM OF FINITE INDUCTION

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A graduate student in mathematics normally uses the Axiom of Finite Induction in proofs over and over again. Familiarity with the Axiom quite often leads to a certain looseness in its application, namely, the student no longer bothers to write down or think carefully about exactly what the induction statement is. The result of this in many cases is that he errs by drawing too strong a conclusion from his proof. The following incorrect proof of the (true) statement that the set P of positive integers is a well-ordered set occurs frequently and illustrates the point:

(1) The set $\{1\}$ is trivially well-ordered.

(2) Assume that the set $S\equiv\{1, 2, 3, \cdots, k\}$ is a well-ordered set.

(3) Let $T\equiv\{1, 2, 3, \cdots, k, k+1\}$ and T' any nonempty subset of T.

Case I. $T'\equiv\{k+1\}$. Then $k+1$ is the least element of T'.

Case II. $T'\not\equiv\{k+1\}$, i.e., $S\cap T'=S'\neq\emptyset$. Now $S'\subset S$; therefore, S' has a least element k' by (2). Since $k'\in S'\subset S$, $k'<k+1$, and k' must also be the least element of T'. Thus T is a well-ordered set, and it follows by induction that P is a well-ordered set.

The final conclusion that "P is a well-ordered set" does not follow from the proof. What does follow is that "for any positive integer n, the set $P_n \equiv\{1, 2, 3, \cdots, n\}$ is a well-ordered set." This is an induction statement about a class of finite sets, and P is not a member of this class. If the final conclusion above *were* correct, one could prove in exactly the same way that a denumerable intersection of (real) open intervals is an open interval, which, of course, is a false statement.

SOME PROOFS OF A THEOREM ON QUADRILATERAL

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Dedicated to the memory of VICTOR THÉBAULT

In this paper I shall give several different proofs of a theorem on quadrilateral related below. Some of the proofs I believe to be new, and the others are taken from eminent books perhaps the readers are familiar with them. I all collect here to make a comparison between them, perhaps it may stimulate the interest of college students or school geometry teacher.

THEOREM. *If a quadrilateral be circumscribed about a circle, the diagonals and the lines joining the opposite points of contact are concurrent.*

Let $ABCD$ be a quadrilateral circumscribed about a circle Σ, and the points of contact on each side are E, F, G and H respectively, as shown in the figure. It is required to prove that AC, BD, EG and FH are concurrent.

For briefness, we adopt the following notations:

$M \cup N$ represents the line passing through M and N.

$AB \cap CD$ represents the intersection of AB and CD.

⫽ represents the homothetic relation, i.e. similar and similarly situated.

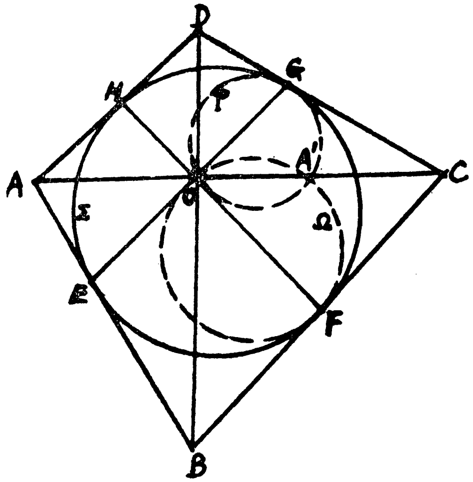

FIG. 1

Proof 1. By the method of inversion. (See fig. 1.)

Invert the figure with O, the intersection of HF and EG, as center of inversion, and the power of O with respect to the circle Σ as the constant of inversion. Then the circle Σ inverts into itself, E and G also F and H are inverse points. The tangent AE inverts into a circle Γ passing through O and touching the circle Σ at G. Similarly the tangent AH inverts into a circle Ω passing through O and touching the circle Σ at F. The intersection A' of these two inverse circles is the inverse of A. $\therefore A$, O, A' are collinear. Since C is the radical center of these three circles Σ, Γ and Ω, hence OA' must pass through C, i.e. AC, EG and FH

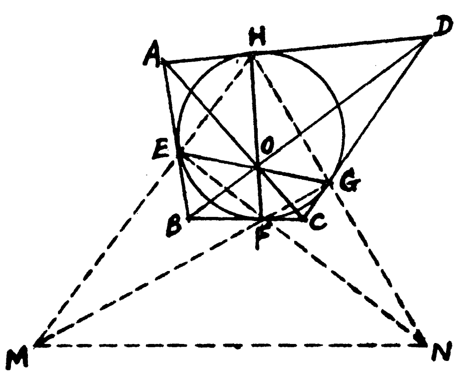

FIG. 2

are concurrent. Similarly, BD, EG and FH are concurrent. $\therefore AC$, BD, EG and FH meet in a point O.

Proof 2. By the theory of pole and polar. (See fig. 2.)

Let $HE \cap FG = M$, $EF \cap HG = N$, $EG \cap FH = O$. Since the polars of A and C, i.e. $E \cup H$ and $F \cup G$, pass through M, so the polar of M is $A \cup C$. Similarly, the polar of N is $B \cup D$. $\therefore M \cup N$ is the polar of $AC \cap BD$. But $M \cup N$ is the polar of O, $\therefore AC \cap BD \equiv O$. Hence AC, BD, EG and FH meet in a point O.

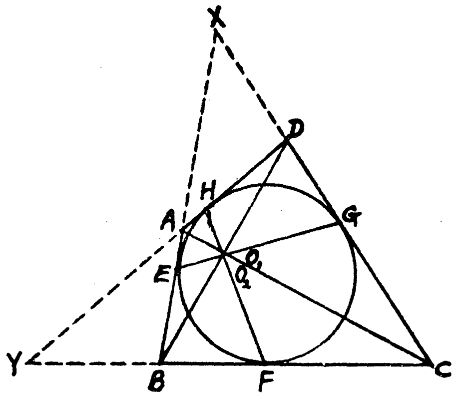

FIG. 3

Proof 3. Alternate method of polar. (See fig. 3.)

Let $AB \cap CD = X$, $BC \cap AD = Y$, $EG \cap FH = O$. Since EG and HF are the polars of X and Y respectively, so $X \cup Y$ is the polar of O. $\therefore Y\{AC, XO\} = X\{AC, YO\} = -1$, $\therefore A, O, C$ are collinear, i.e. AC, EG and FH are concurrent. It is evident that AC, BD, EG and FH meet at O.

Proof 4. Derived from Brianchon Theorem. (See fig. 3.)

Regard $AEBCGD$ as a circumscribed hexagon, by Brianchon Theorem we know at once that AC, EG and BD are concurrent. Similarly, if we regard $BFCDHA$ as a circumscribed hexagon, we know BD, FH and AC are concurrent. $\therefore AC$, BD, EG and FH are concurrent.

Proof 5. By the theory of transversals. (See fig. 3.)

Let $AB \cap CD = X$, $BC \cap AD = Y$, $EG \cap AC = O_1$, $HF \cap AC = O_2$. Now it is required to prove that $O_1 \equiv O_2$.

Regard EG as a transversal of $\triangle XAC$, then we have

$$\frac{AO_1}{O_1C} \frac{CG}{GX} \frac{XE}{EA} = 1, \qquad \therefore \frac{AO_1}{O_1C} = \frac{EA}{CG}$$

Next, regard HF as a transversal of $\triangle YAC$, then we obtain

$$\frac{AO_2}{O_2C} \frac{CF}{FY} \frac{YH}{HA} = 1, \qquad \therefore \frac{AO_2}{O_2C} = \frac{HA}{CF}.$$

But

$$HA = EA, \quad CF = CG, \quad \therefore \frac{AO_1}{O_1C} = \frac{AO_2}{O_2C}, \quad \therefore O_1 \equiv O_2. \quad \therefore \text{\&c.}$$

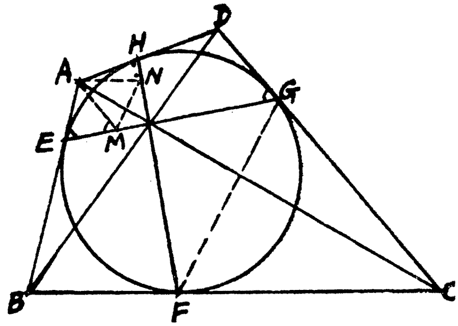

FIG. 4

Proof 6. By the theory of homothecy. (See fig. 4.)

Draw $AM \| DG$, $AN \| BF$, cutting EG and HF at M and N respectively. Join MN and FG. Then $\angle AEG = \angle DGE = \angle AME$. $\therefore \triangle AEM$ is isosceles. Similarly, $\triangle AHN$ is isosceles. But $AE = AH$. $\therefore AM = AN$, and the $\triangle AMN$ is isosceles. Also the $\triangle CFG$ is isosceles. Now the two isosceles triangles AMN and CGF having two sets of equal sides are parallel and in opposite direction (i.e. $AM \| CG$, $AN \| CF$), therefore their bases MN and GF are also parallel and in opposite direction. $\therefore \triangle AMN \not\sim \triangle CGF$. $\therefore AC$, EG, and HF are concurrent. Similarly, BD, EG and HF are concurrent. $\therefore$ &c.

Now, I give some proofs more elementary as follows:

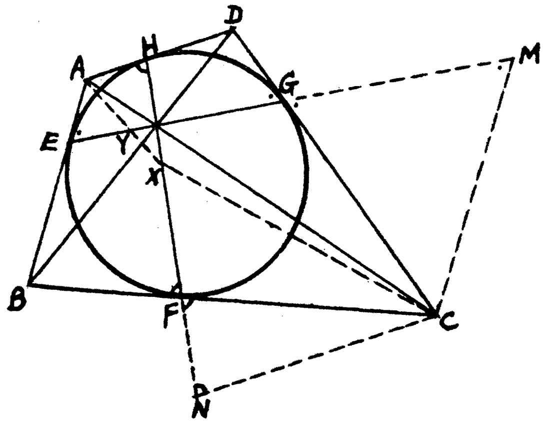

FIG. 5

Proof 7. Apply the theorem on similar figures. (See fig. 5.)

Draw $CM \| AB$, $CN \| AD$, cutting EG and HF at M and N respectively. Then $\angle M = \angle AEY = \angle DGE = \angle CGM$, $\therefore \triangle CMG$ is isosceles. Similarly, $\triangle CNF$ is isosceles. But $CG = CF$. $\therefore CM = CN$. Now, if AC do not pass through the intersection of EG and HF, let $AC \cap FH = X$, and $AC \cap EG = Y$, then

$$\frac{AE}{AY}=\frac{CM}{CY}, \quad \text{and} \quad \frac{AH}{AX}=\frac{CN}{CX}\cdot$$

But $AE=AH$, therefore we have

$$\frac{AX}{CX}=\frac{AY}{CY}, \qquad \therefore X\equiv Y\equiv HF\cap EG.$$

Hence AC, EG and HF are concurrent. Similarly, BD, EG and HF are concurrent. ∴&c.

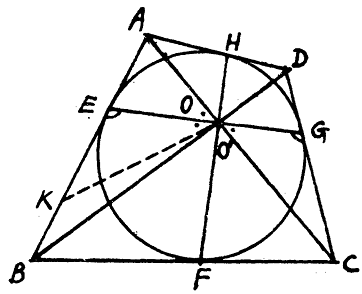

FIG. 6

Proof 8. Alternate method of similar figures. (See fig. 6.)

Let $AC\cap EG=O$, on EB take a point K, so that $\angle EOK=\angle GOC$, now because

$$\angle OEK=\angle OGC, \quad \therefore \triangle OEK \backsim \triangle OGC, \quad \therefore \frac{OK}{KE}=\frac{OC}{CG}\cdot \tag{1}$$

But

$$\angle AOE=\angle COG=\angle EOK, \qquad \therefore \frac{OA}{AE}=\frac{OK}{KE}\cdot \tag{2}$$

From (1) and (2) we have

$$\frac{OA}{AE}=\frac{OC}{CG}, \qquad \therefore \frac{AO}{OC}=\frac{AE}{CG}\cdot \tag{3}$$

Similarly, let $AC\cap HF=O'$, then

$$\frac{AO'}{O'C}=\frac{AH}{CF}\cdot \tag{4}$$

But $AE=AH$, $CG=CF$, hence, from (3) and (4), we obtain

$$\frac{AO}{OC}=\frac{AO'}{O'C}, \qquad \therefore O\equiv O'.$$

Hence we know that the three lines AC, EG, HF meet at O. ∴&c.

Proof 9. Apply the theorem on the ratio of areas. (See fig. 6.)

Let $AC \cap EG = O$. Since $\angle AOE = \angle COG$, and $\angle AEO + \angle CGO = 2$ rt $\angle$s, therefore

$$\frac{\triangle AEO}{\triangle CGO} = \frac{AE \cdot EO}{CG \cdot GO} = \frac{AO \cdot OE}{CO \cdot OG},$$

hence

$$\frac{AO}{OC} = \frac{AE}{CG}.$$

Similarly, if $HF \cap AC = O'$, we have

$$\frac{AO'}{O'C} = \frac{AH}{CF}.$$

But $AE = AH$, $CG = CF$, therefore we obtain

$$\frac{AO}{OC} = \frac{AO'}{O'C}, \quad \text{i.e.} \quad O \equiv O'.$$

Hence AC, EG, HF meet in a point O. ∴&c.

NEW ROLES FOR OLD FIGURES

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Introduction. Advancements in science and technology have been so numerous in recent years that we are having to gallop in order to stand still—to paraphrase the mathematician, Lewis Carroll. To help the scientist in solving the complex problems of automation, disease, space travel, etc., high-speed electronic computers have been developed. In fact, the world of tomorrow seems destined to be controlled by computers, as already we have computers that regulate highway traffic, measure the amount of smog in the atmosphere, help the heart specialist diagnose diseases of the heart and make airline and theatre reservations.

In order that the modern large-scale digital computer may perform such a wide variety of tasks, it is necessary to employ a number of special techniques. These techniques have resulted in the appearance of new topics in mathematics —assigning, as it were, new roles to our familiar numbers. It is with these new mathematical ideas that this paper is concerned.

The Digital Computer. In spite of the apparent complexity of a digital computer, it can do only what its name implies—it can count. Counting forward is the same as adding, counting backward is the same as subtracting. If we can add, we can also multiply since multiplication is successive addition. In a like manner, division is successive subtraction. Thus by merely counting, a digital computer can add, subtract, multiply and divide; in short, it can do simple arithmetic. Now we are getting to the heart of the matter; how can a computer, which can do only simple arithmetic at the fifth grade level, solve complex scientific problems? The answer lies in the fantastic speeds at which our computers operate and in the uses we have made of our mathematics. We have had to shift emphasis and we have had to develop such mathematical techniques as are necessary.

Let us consider some examples of special techniques in mathematics, techniques which are especially useful for high-speed computers. First, a method for finding the square root of a number. Why is $\sqrt{25}=5$? Because $25-1-3-5-7-9=0$, and we have had to subtract the first *five* odd numbers in order to arrive at zero. Here we take advantage of a theorem in algebra which can be proved by induction, namely, that the sum of the first n odd numbers is n^2. Thus, $\sqrt{15130}$ can be approximated by 123 since the sum of the first 123 odd numbers is 15129. Hence we can calculate $\sqrt{1.5130}$ as 1.23 with an error in its square of 0.0001.

Another method that is somewhat unique is known as the "relaxation" method. Consider as an example of this method the pair of equations

$$\begin{cases} 79 - 3x + y = 0 \\ 147 + x - 3y = 0 \end{cases}$$

which we wish to solve for x and y. These equations may be written in the form

$$R_1 = 79 - 3x + y$$
$$R_2 = 147 + x - 3y$$

where, for the correct values of x and y, $R_1=R_2=0$. For any other values of x and y, R_1 and R_2 are *not* zero. R_1 and R_2 are called the "residuals" and are a measure of the error of the solution for any given x and y. R_1 and R_2 can be calculated for any pair of x and y, but we want to find *that* unique pair which makes R_1 and R_2 zero *simultaneously*. We begin by making a table in which we enter our guesses for x and y. They are exactly that for it is absolutely immaterial what values we use initially since we will eventually obtain the correct values. The method gets its name not because *we* have relaxed but because we have relaxed the residuals until they became zero.

RELAXATION TABLE

Step No.	x	y	R_1	R_2
1	$x=0$	$y=0$	79	147
2		$\Delta y=49$	128	0
3	$\Delta x=42$		2	42
4		$\Delta y=14$	16	0
5	$\Delta x=5$		1	5
6		$\Delta y=1$	2	2
7	$\Delta x=1$		1	3
8		$\Delta y=1$	0	0
	$x=48$	$y=65$		

Check: $R_1= 79-144+ 65=0$
$R_2=147+ 48-195=0$

There is a systematic method of bringing the residuals to zero. The largest residual is reduced first as shown in the Relaxation Table. In the example $R_2 = 147$ and since this comes from the second equation, it can be reduced to zero by increasing y by an amount $\Delta y = 49$. Since y has a coefficient -3 in the second equation, the effect is to reduce y by 3(49) or 147 bringing R_2 to zero. This change in y also changes R_1 and increases it 128. Now the 128 can be reduced to 2 by changing x by an amount $\Delta x = 42$ so that the -3 coefficient of x in the first equation reduces R_1 by an amount 3(42) or 126 leaving a residual $R_1 = 2$. This last change increases R_2 to 42 and next this value is reduced. The process continues in this manner until R_1 and R_2 are both zero. If integral changes in x and y are not enough to complete the relaxation process, then fractional changes can be made and the final values of x and y can be found to any desired degree of accuracy.

It should be noted that the changes in x and y, the Δx and Δy in the Relaxation Table, need not always be positive. If the first change in x had been 50 instead of 42, it would have been necessary to have a negative Δx at some later stage. Making a greater change than necessary is called "over-relaxation" and is used to speed up the convergence process. The relaxation method of solving simultaneous linear algebraic equations is of value in teaching students the significance of "convergence" and "speed of convergence" as well as demonstrating that the final result can be reached by many different paths. Problems of this type can be used in competitions in the classroom also. Students may compete against each other to see who can arrive at the answer by the least number of steps.

Another interesting digital technique is one called the Monte Carlo method for solving partial differential equations. Suppose we have a region bounded by the curve C and suppose that we know the temperature at many places along this boundary. Then the problem is to find the temperature, T, at an arbitrary point P, within the region. Starting at the point P we make a random walk, i.e., we take steps of length δS and in a positive or negative direction parallel to the coordinate axes. It is apparent that if we make a large number of these random walks we will arrive at the boundary nearest the point P more often than at some other point on the boundary.

This is simply another way of saying that the temperature at P is influenced more by the temperature at the boundary nearest to P than by other points on the boundary. This is exactly what we mean when we say we are solving the partial differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

subject to given boundary conditions. Mathematically, if T_{c_i} denotes the temperature at the point on the boundary reached by the ith random walk, then the temperature at P, T_P, is given by

$$\operatorname*{Limit}_{n \to \infty} \sum_{i=1}^{n} T_{c_i}/n = T_P$$

where n denotes the total number of random walks originating at the point P. It can be seen that the temperature at various points of interest in the region can be found by this method. There have been published reliable lists of random numbers which can be used to form the necessary paths of the random walks.

The Monte Carlo example is a good way to introduce the concepts of randomness and probability. It is, of course, not necessary to say anything about partial differential equations. The "random walk" technique is useful in many fields and a knowledge of its principles can give the student an appreciation for such topics as random sampling in public opinion polls, some of the methods for testing for randomness, and the genuine uses of probability in mathematics.

The purists may disdain these new methods and prefer the former direct approach to these "beating-around-the-bush" techniques, but greater insight into mathematics can result from an understanding of the new methods. The Monte Carlo method just described can be used to teach the concept of what a solution to Laplace's equation really means to students who have not had advanced mathematics or physics.

Summary. We have attempted to show that modern technology requires that some of our teaching methods need to be changed. This is especially true in mathematics where it has become necessary to revise the course content to conform to the automation that has come to arithmetic in the form of computers. Our way of life is being changed by the ever-increasing use of computers ranging from the circuitry of the automatic washer to the huge electronic system used to track artificial satellites. The applications of computers are many and come from many different fields. Railroads are using computers to switch their cars; the home economist uses a computer to determine the types of food to buy to obtain the necessary nutrients at minimum cost; the social scientists are using computers to solve certain decision-making problems in the political field and so on, practically without end.

In conclusion we should stress that automation will never *lower* the dignity of man. He has to be more clever than before, he has to use his intellect more than ever. He has to be the greatest of teachers because he has to express the most complicated ideas in the simplest terms. He has to devise means of trading great intellect and slow speed for low intellect and high speed and make a profit! In his ability to do this he will improve his lot and rise to greater heights using machines as stepping stones. Man's heart and mind will continue supreme over any mechanical device.

DIFFICULTIES IN PROPOSING UNDERGRADUATE RESEARCH*

BROTHER T. BRENDAN, F.S.C., Saint Mary's College, California

In the spring of 1961 Professor Donald Western circulated a report to Association members calling attention to the paucity of proposals made to the undergraduate research program of NSF. This was followed by a five-day conference devoted to the problem and supported by NSF at Carleton College in

* These remarks are based on a talk given November 18, 1961, at the annual convention of the Northern California Mathematics Council, College Section, Asilomar.

June, 1961. Perhaps the chief concern of this conference was that of trying to define "research" in a pure enough way to be acceptable to top-flight mathematicians and at the same time in a realistic enough way to be admissible of undergraduate participation. It was the lack of such a definition which seemed responsible for the hesitation shown by undergraduate departments in making proposals to NSF. Of course, many of these departments were undoubtedly carrying on genuine research with undergraduates at their own expense.

About sixty college teachers were in attendance at the conference. The results of their discussions among themselves, and with the experts called in, are well presented in the official Report prepared by Professors K. O. May and S. Schuster. It was mailed to all the colleges and universities in the United States during November and December, 1961, and so should be available to most of the readers of this journal.

It seems to me, by hindsight, that there were several matters which those of us who were participants in the conference failed to discuss or failed to discuss adequately. This hindsight was reinforced in October, 1961, when several of the conference members were asked by Science Service, working for NSF, to sit on review panels. One group met in San Francisco and another in Washington, D. C., to evaluate fifty proposals (as against eighteen the year before) in undergraduate mathematics research and independent study. Two officials of NSF, Dr. James S. Bethel and Dr. Lewis Pino, were present and urged us to continue to thrash out the problems raised at the conference.

As a result of the added experience of being a panelist, I would say that there remain three unsolved problems: Should the undergraduate research program try to support proposals that define research at a level as low as that tolerated in other disciplines? Can the NSF program be expected to help those colleges that are too understaffed to guide genuine research and independent study? How does one recognize "deadend" research?

1) Suppose a proposal is made which involves using undergraduates to do quite a bit of leg-work in the gathering of statistics and then to help in processing them. Granting the adequacy of its supervising staff and granting all the other prerequisites, should NSF (or the department itself) support such a proposal in the name of research? Many other disciplines, such as psychology or political science, might regard such an exercise as valuable preparation for graduate work. Does not mathematics too, at least in one or two domains of its interest, require a gathering of data as part of research? The answer to this general problem, of course, depends on what we want undergraduate research in mathematics to do. It presumably does several things—stimulates interest, rewards the diligent, etc. But its chief purpose appears to be preparation for graduate school—that is, an initiation into some of the skills and techniques which will help make the transition to graduate school smoother. By and large most of us have an image of graduate school, and a notion of its skills, which preclude proposals that are "impure." May not our judgment be based on a false dichotomy? Certainly both pure and applied mathematicians appreciate each other's problems and both require essentially the same background.

2) By and large, one gets the impression that, with NSF, proposals from small colleges do not stand much chance as against those from large universities.

This is mostly because small colleges cannot provide a competent staff at all, or cannot free busy members of its competent staff even for that fraction of time necessary to supervise a proposed study. Money does not always buy time. As presently designed the NSF program cannot help such colleges. What is needed, perhaps, is a re-designing of the program or a new and different program. I know that some panelists made specific suggestions along these lines, but there is obviously room for much more thought in this direction.

3) What about proposals designed to take some gifted students through certain parts of nineteenth century mathematics which today seem deadended? This raises the problem of the definition or even the possibility of deadend research. At any moment results of past research may become highly relevant to some area of practical or theoretical concern and thus become openended. So that it may well be that there is no such thing as deadend research. Perhaps it is not wise to isolate our students or ourselves from what appears to be the completed research of the past. If a professor has an enthusiasm, even though it is for an area of classical mathematics, should he not be encouraged to communicate that enthusiasm despite the fact that it is not directly geared to present-day graduate study? It seemed generally agreed at the Carleton conference that *re*discovery has a psychological and educational value nearly as great as first discovery.

At this point I would like to insert a parenthetical remark. It has been called to my attention that even though NSF offers to support undergraduate research and independent study in the *history* of science and mathematics, no proposals specifically in this area were received in 1961. And very few were in the area of logic. It would seem that both these fields would be "naturals" for undergraduates.

For me, however, none of the above difficulties constitutes the main one. My problem, and I suspect that it is shared by many college teachers and may be partly responsible for the paucity of proposals to NSF, is that the Foundation's aims may not be compatible with the college's aims. To put the matter succinctly, an undergraduate liberal arts college is non-professional or at least pre-professional; the NSF program is clearly professional. Unless a college is committed to a kind of professionalism, then, it runs a risk in proposing undergraduate research. Of course, if it takes the precaution of selecting the right kind of student, then a minimal amount of harm, and a lot of good, may result. But we can easily choose the wrong kind. It seems to me that the wrong kinds of student are two: the ones too weak ever to do well in graduate school, and the ones too strong and too eager. It is the latter whom I worry about. If we have an especially bright and enthusiastic student in mathematics who is already set for graduate school and almost certain to do well there, we do him a disservice when we encourage his appetite for narrow specialization. Our duty as college teachers at the undergraduate level is to make such a student realize that his precious undergraduate years—the last leisure he may ever have for diversifying his interests—are slipping away, and that he should beware of undertaking time-consuming graduate work too soon. As one young Ph.D. in physics told me recently, "In graduate school red-hots are a dime a dozen; educated ones are not."

MISCELLANEOUS NOTES

EDITED BY ROY DUBISCH, University of Washington

Articles intended for this department should be sent to Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.

ON SCHWARZ'S INEQUALITY

SEYMOUR GOLDBERG, New Mexico State University

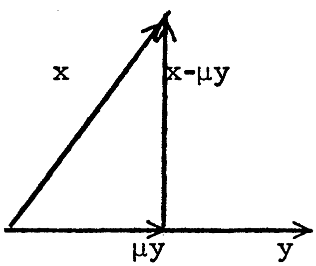
In Euclidean 3-space, designated by E^3, the inner product, $\langle x, y\rangle$, of two vectors x and y is usually defined by $\langle x, y\rangle = \|x\|\ \|y\| \cos\theta$, where $\|x\|$ and $\|y\|$ are the lengths of the vectors and θ is an angle between them. One then proceeds to show that for $x=(x_1, x_2, x_3)$ and $y=(y_1, y_2, y_3)$, $\langle x, y\rangle = x_1y_1+x_2y_2+x_3y_3$. In particular, $\|x\|^2=\langle x, x\rangle$. This concept of an inner product is extended in a natural manner to vector spaces over the real or complex numbers. For a discussion of such spaces, the reader is referred to most any linear algebra text e.g., [1], chapter 8.

The student who is introduced to the concept of an inner product space X for the first time is invariably shown by the instructor the close relationship between E^3 and X. For example, in E^3 we have that $|\langle x, y\rangle| = \|x\|\ \|y\|\,|\cos\theta| \leqq \|x\|\ \|y\|$. In X the same inequality, called Schwarz's inequality, holds where $\|x\|^2=\langle x, x\rangle$. The usual proof given (see for example [1] p. 228) is to expand $\langle x-\lambda y, x-\lambda y\rangle$ thereupon choosing λ so that "everything comes out just right." However, from the standpoint of the student, this proof only involves manipulation which gives no geometrical insight as in E^3.

It is the purpose of this note to give a geometrically motivated proof of Schwarz's inequality which is essentially the same as the one given in E^3.

Schwarz's inequality: Let X by a inner product space with inner product $\langle\ ,\ \rangle$. Then $|\langle x, y\rangle| \leqq \|x\|\ \|y\|$. If equality holds, then x and y are linearly dependent.

Proof. The first step is to give meaning to $\cos\theta$ for $X \neq E^3$. To avoid trivialities, it is assumed in the sequel that x and y are both non-zero vectors in X.



As in the figure, choose μ so that $x-\mu y$ is perpendicular to y, i.e. $0=\langle x-\mu y, y\rangle$, or equivalently, $\mu=\langle x, y\rangle/\|y\|^2$. Define $\cos\theta$ and $\sin\theta$ by $\cos\theta=\|\mu y\|/\|x\|$, $\sin\theta = \|x-\mu y\|/\|x\|$.

By expanding $\langle x-\mu y, x-\mu y\rangle$ we obtain $\|x-\mu y\|^2=\|x\|^2-\|\mu y\|^2$ from which it follows that

$$(1) \qquad \cos^2\theta + \sin^2\theta = 1$$

whence $0 \leqq \cos\theta \leqq 1$. Thus

$$\|x\|\|y\| \geqq \|x\|\|y\| \cos\theta = |\langle x, y\rangle| .$$

Suppose that $|\langle x, y\rangle| = \|x\| \|y\|$. Then $\cos^2\theta = |\mu|^2\|y\|^2/\|x\|^2 = |\langle x, y\rangle|^2\|y\|^2\|x\|^2 = 1$. But then by (1), $0 = \sin\theta = \|x - \mu y\|/\|x\|$, or $x = \mu y$.

Reference

1. K. Hoffman and R. Kunze, Linear algebra, Prentice Hall, 1961.

A COMPLEMENTARY PROBLEM ON NONPLANAR GRAPHS

FRANK HARARY,* The University of Michigan

It has been found empirically by John L. Selfridge, in work with networks to be used as printed circuits, that for every network with nine nodes which was encountered, either it or its complementary network could not be printed with no intersecting arcs. In terms of graph theory this conjecture asserts that for every graph G with $p = 9$ points, either G or its complementary graph $\overline{G}$ is nonplanar. For a description of graphs and planar graphs, see the books by Berge [1] and König [4] or the article [2].

If this conjecture holds for graphs with 9 points, it clearly also holds for graphs with $p > 9$ points. It was remarked in the problem, [3], that a simple argument using Euler's polyhedron formula readily demonstrates the conjecture for all graphs having $p \geqq 11$ points. The purpose of this miscellaneous note is to supply that argument. For clarification we begin with a few diagrams.

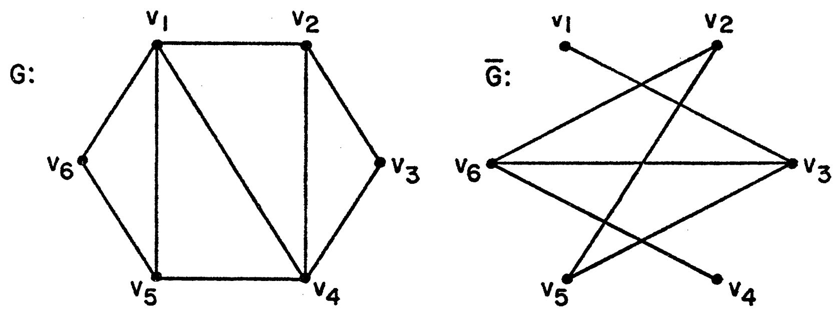


FIG. 1

In Fig. 1, we show a graph G with $p = 6$ points and $q = 9$ lines, and its complementary graph $\overline{G}$ with the same 6 points but containing those 6 lines which join pairs of points not adjacent in G.

Note that the graph $\overline{G}$ of Fig. 1 is planar since it can be redrawn in the plane as in Fig. 2 with no two of its arcs intersecting.

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* This note is RAND paper P-2534, February 1962.

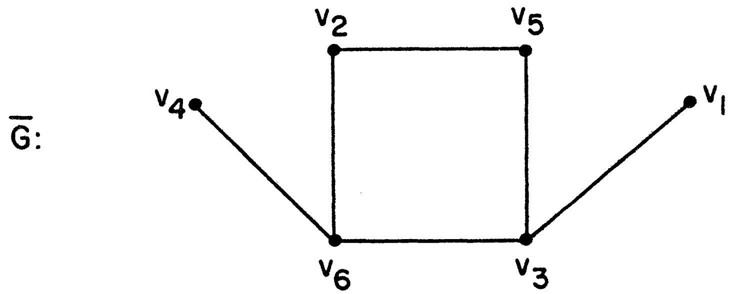


FIG. 2

In Fig. 3, the graph H and its complement $\bar{H}$ are "isomorphic." Such a graph is called *self-complementary*.

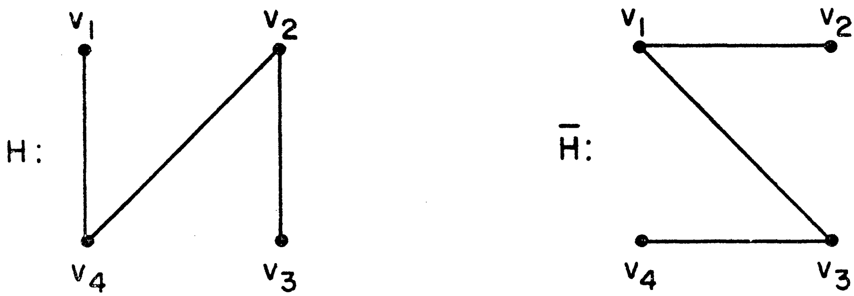


FIG. 3

Finally Fig. 4 shows a planar graph in which every region, including the exterior, is triangular. Such a graph is called *a triangulation of the 2-sphere.*

The fact that the conjecture is false for $p=8$ may be easily verified by the reader. For there is a planar graph with 8 points which is both a triangulation of the 2-sphere and is self-complementary.

Let G be a graph with p points and q lines.

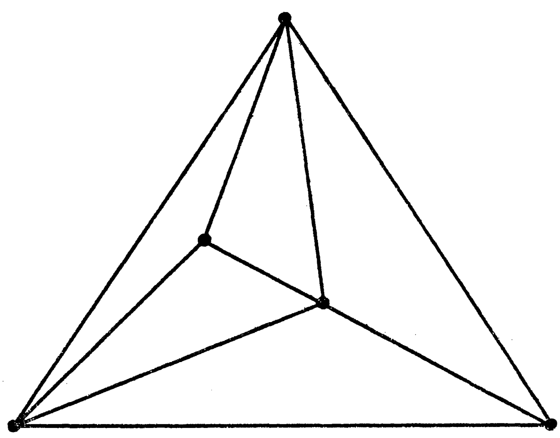


FIG. 4

THEOREM 1. *If G is a triangulation of the 2-sphere, then $q=3p-6$.*

Proof. By Euler's Polyhedron Theorem, $V-E+F=2$ for G regarded as a spherical polyhedron, where V, E, and F are the number of vertices, edges, and faces respectively. Since $V=p$ and $E=q$, we only need to find an expression for F. As illustrated in Fig. 4, every edge lies in exactly two faces (triangles) and every face contains three edges; hence $3F=2E$. Substituting $F=2q/3$, we find that $p-q+2q/3=2$ so that $q=3p-6$.

COROLLARY 1. *If G is a graph with $q>3p-6$, then G is nonplanar.*

Proof. On adding any one new arc (joining two points not already adjacent) to a triangulation of the 2-sphere, this arc will necessarily intersect at least one old arc.

COROLLARY 2. *If G is a graph with at least 11 points, then G or $\bar{G}$ is nonplanar.*

Proof. Let G have $p\geqq 11$ points and q lines, and let $\bar{G}$ have $\bar{q}$ lines. Then $q+\bar{q}=p(p-1)/2$. Hence the average value of q and $\bar{q}$ is $p(p-1)/4$. Thus $q\geqq p(p-1)/4$ or $\bar{q}\geqq p(p-1)/4$. For $p=11$, $p(p-1)/4=27.5$. But for $p=11$, $3p-6=27$. Applying Corollary 1, the present corollary is proved for $p=11$. The proof for $p>11$ is the same.

For $p=9$ and $p=10$, the conjecture remains unsolved. My own guess is that it is also true for these values of p. Some of my best friends have attacked this problem. They report that every triangulation with $p=9$ of the 2-sphere which they have constructed has a nonplanar complement. A brutal affirmative solution could be accomplished by proving that every such possible triangulation has a nonplanar complement.

Added in proof: Since this manuscript was submitted for publication, the conjecture has been proved for $p=9$ (and hence for $p=10$) independently three times. First, we (J. Battle, F. Harary and Y. Kodama, Every planar graph with nine points has a nonplanar complement, Bull. Amer. Math. Soc., to appear Nov. 1962) verified the conjecture using Kuratowski's Theorem characterizing planar graphs, the partition of a graph, and an exhaustive listing of cases of 9 vertex triangulations of the sphere with given partitions. Then John R. Ball of the Carnegie Institute of Technology verified it by a somewhat similar exhaustive proof. Finally W. T. Tutte employed precisely the brutal method mentioned in the last paragraph of the note, taking only two days for the job. No elegant proof of the result is known.

References

1. Berge, C., Theorie des graphes et ses applications, Paris, 1958.
2. Harary, F., Some Historical and Intuitive Aspects of Graph Theory, SIAM Review, 2 (1960) 123–131.
3. ———, "Problem 28," Bull. Amer. Math. Soc., 67 (1961) 542.
4. König, D. Theorie der endlichen und unendlichen Graphen, Leipzig, 1936; reprinted New York, 1950.

EDITED BY HOLBROOK M. MACNEILLE, Case Institute of Technology

This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to Holbrook M. MacNeille, Department of Mathematics, Case Institute of Technology, Cleveland 6, Ohio.

FURTHER PROPERTIES OF THIRD ORDER DETERMINANTS

JOEL E. COHEN, Harvard University

C. W. Trigg (MATHEMATICS MAGAZINE 35: 78, 1962) presents the following property of third order determinants and their related "twists:"

$$(1)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ b_2 & c_2 & a_2 \\ c_3 & a_3 & b_3 \end{vmatrix} = 0.$$

A different, but equivalent, representation of this result reveals an underlying symmetry which suggests two further analogous properties.

Let $V_i=(a_i,\ b_i,\ c_i)$, $i=1,\ 2,\ 3$. Define $s^0V_i=V_i$, $s^1V_i=(c_i,\ a_i,\ b_i)$, $s^2V_i=(b_i,\ c_i,\ a_i)\ \cdots$. Then (1) takes the form

$$(2)\qquad V_1\cdot(V_2\times V_3)+V_1\cdot(s^1V_2\times s^2V_3)+V_1\cdot(s^2V_2\times s^1V_3)=0,$$

where "$\cdot$" means scalar product and "$\times$" means vector product. The left member of (2) equals $V_1\cdot(V_2\times V_3+s^1V_2\times s^2V_3+s^2V_2\times s^1V_3)$, and the right hand factor of this product must be 0, since (2) holds for all V_1. For arbitrary $(V_2,\ V_3)$, define

$$[s^0\times s^0+s^1\times s^2+s^2\times s^1](V_2,\ V_3)=V_2\times V_3+s^1V_2\times s^2V_3+s^2V_2\times s^1V_3.$$

Then

$$(3)\qquad s^1\times s^2+s^0\times s^0+s^2\times s^1=0,$$

i.e., the left member annihilates every pair of 3-vectors. The form of (3) suggests that

$$(4)\qquad s^0\times s^2+s^1\times s^1+s^2\times s^0=0$$

and

$$(5)\qquad s^0\times s^1+s^2\times s^2+s^1\times s^0=0,$$

both of which may be easily verified.

These results also hold for vertical permutations. If $*V$ is the column vector which is the transpose of V, and W is a column 3-vector, define $tW={}^*s^{*^{-1}}W$. Then we may substitute t for s in (3), (4), and (5), since the value of a determinant is not affected by interchanging rows and columns.

The Solution of Equations in Integers. By A. O. Gelfond (translated from the Russian and edited by J. B. Roberts.) Freeman Publishing Company, San Francisco, 1961, viii+63 pp., $1.00.

The original Russian edition of this book appeared in 1957 and was based on a lecture given by Prof. Gelfond at Moscow State University in 1951. This translation is one of the Golden Gate series of paperbacks. While it will undoubtedly prove a useful introductory text at the undergraduate level, it should be made clear that there is ample evidence of the origin of the work in a lecture, and that there is no pretence of completeness in the coverage of the subject. In fact, it might have been more appropriate to have the words "Lecture" and "Some Equations" in the title.

In a brief historical note erroneous dates are given for Euclid and Diophantus, but whether these errors are historical or typographical is not certain. The main part of the work deals with the complete solution of $ax+by+c=0$ and $x^2-Ay^2=1$, for which continued fractions are introduced, and with Fermat's Last Theorem, where the insolubility in integers of $x^4+y^4=z^2$ is proved. It is stated that Fermat's Last Theorem has been proved for any exponent divisible by a prime less than 100, but in fact it is now known to be true for primes up to 690 at least. In addition a number of related general cases as well as some particular equations are dealt with. The chief omission is the theory of congruences: while use is made of remainders on division by 4, the formal concepts and symbolism of congruence are not introduced. Clearly this is quite acceptable in the course of a lecture, but whether it is so in even a short book is not so certain.

The style is clear, but by no means perfect: as an example I would mention that the solution $x^2+y^2=z^2$ is given in one form when later it is required in another form. In the one-page index I noted nine mistakes or irregularities.

ALAN SUTCLIFFE
Knottingley, Yorkshire, England

Introduction to Linear Programming. By Walter W. Garvin. McGraw-Hill, New York, 1960, 281 pp., $8.75.

According to the author's preface, "A large part of the material in this book was prepared originally in the form of lecture notes for a course given to selected personnel of the Standard Oil Company of California. The purpose of this course was to acquaint company scientists, engineers, and economists with linear programming and to teach them how to use this new analytical tool and how to recognize and formulate linear-programming problems. In view of the heterogeneous mathematical background of the participants, the course avoided matrix notation and did not employ any logical reasoning that went beyond elementary algebra and simultaneous linear equations. This level was maintained during the conversion of the original lecture notes into the present text."

This reviewer found the book extremely well written and easy to read. The author does a commendable job of staying within his stated objectives of avoiding other than elementary mathematics. Each new idea is first introduced with reference to its contribution to a practical problem or to improving some previously developed approach and, then, is followed by a lucid theoretical explanation and well chosen numerical examples. Flow diagrams are used by the author to assist in the explanation of the logic of computational procedures. Worthy of particular mention is the material on sensitivity analysis, linear programming with upper bounds, resolution of degeneracy, parametric linear programming, the generalized transportation problem, linear programming under uncertainty, dual ity, and the revised simplex method. Such material has not previously been brought together in a work of this kind, and the literature dealing with it generally requires mathematical sophistication beyond that required to understand this book. The level and clarity of this book should make it an excellent reference book for scientists with limited mathematical training or an excellent text for undergraduate science or graduate business school introductory courses on linear programming.

The book is divided into three parts. Part I comprises five chapters. Chapter 1 deals with a definition of the general linear programming problem, and proves three fundamental theorems necessary for understanding linear programming. Chapter 2 explains the simplex method, and Chapter 3 describes the computational procedure for the simplex method. Chapter 4 provides a good discussion of sensitivity analysis which considers the effects of changes in the optimum solution caused by changes in the cost coefficients, in the coefficients of the constraint equations, or by the addition of new variables or new constraints. In Chapter 5 a simplified gasoline-blending problem is defined and solved in detail using the techniques developed in the proceeding chapters.

Part II comprises five chapters on the transportation problem and its variants. Chapter 6 discusses the transportation problem for a balanced system. Chapter 7 deals with an unbalanced system and the transportation problem with transshipment. Chapter 8 shows the assignment problem and the caterer problem to be equivalent problems. Chapter 9 describes the tanker-routing problem and demonstrates a method for solving it. Chapter 10 considers the generalized transportation problem and the methods for handling it.

Part III entitled "Special Methods" comprises a chapter on each of the following: linear programming with upper bounds, statistical linear programming, the revised simplex method, resolution of degeneracy, parametric linear programming, a simple economic model, duality, and the warehouse problem.

EDUARDO I. PINA
The Boeing Company

Board and Table Games from Many Civilizations. By R. C. Bell. Oxford University Press, New York, 1960, xxiv+208 pp., hard cover, \$5.00.

Ninety-one games are described, the oldest of which was played some 5,000 years ago. Four of them are described for the first time in English. The games are arranged in six categories, race games (such as backgammon), war games (for example, draughts), morris games, mancala games (such as wari), dice games, and domino games. Each group is given a historical evolutionary treatment interspersed with anecdotes. The rules of the games are exhibited and the discussions are illuminated by about 200 figures and plates. Many of the illustrations depict boards and pieces which have been collected by the author. There are also an excellent chapter on Making Boards and Pieces and an appendix of ten biographies of writers on games. A short glossary, a combined table of contents and bibliography, and an extensive index add to the utility of the volume.

CHARLES W. TRIGG
Los Angeles City College

Arithmetic in Maya. By George I. Sánchez. Privately printed, 2201 Scenic Drive, Austin, Texas, 1961, viii+74 pages, hard cover, $5.00.

The three symbols in Maya vigesimal arithmetic were the shell or closed fist representing zero, a dot for one, and a bar for five. The author shows that in this positional system, addition, subtraction, multiplication, and division could have been performed easily without memorizing multiplication tables through 19 by 19. Calendrical computation is discussed and an abacus that could have speeded up the computations is described. A brief bibliography is appended.

CHARLES W. TRIGG
Los Angeles City College

Proceedings of the Eleventh Symposium in Applied Mathematics of the American Mathematical Society, Edited by Birkhoff and Wigner, American Mathematical Society, Providence, R. I., 1961, v+339 pp., $8.70.

The editors in their preface to the Proceedings make a statement which, unfortunately, applies to many areas of physical research in both theory and experiment; namely, that " . . . very few research mathematicians have, so far, devoted serious efforts to the mathematical problems of nuclear reactor theory." The symposium from which the Proceedings derived had as its goal the demonstration to the mathematics research community the potential richness of reactor theory as both a subject for fundamental mathematical research and, as a consequence, a source of new fundamental concepts. The Proceedings consist of a collection of nineteen individual papers, each written by a recognized authority in reactor theory, and selected so as to cover all the major topics of contemporary reactor theory.

For this reason, the work under discussion offers a refreshing and stimulating experience for the scientist or technologist who is embroiled in conventional reactor theory and calculations as well as for the mathematicians to whom it is addressed. The volume should be of particular interest to the physical theorist who has often had moments of doubt about the validity of his assumptions made "on physical grounds." The novice, on the other hand, will find that the volume offers a succinct description of most of the interesting frontiers of reactor theory, although he may find the density of information somewhat overwhelming.

The introductory paper on reactor types (Weinberg) is an excellent summary that places the details of reactor theory into the proper relation with each other and the field as a whole. All the papers are, of course, written by recognized authorities in the field and hence furnish in one volume a valuable summary of the progress attained thus far in supplying a rigorous foundation for the more mundane but essential chores required for specific analyses. The articles on thermalization (Nelkin) and on resonance absorption (Nordheim) are particularly helpful in this respect.

The contributors and editors have done a most creditable job of demonstrating the mathematical richness of nuclear reactor theory. This reviewer shares their expressed hope that research mathematicians will accept the challenging invitation thus offered.

E. LEIGH SECREST
General Dynamics, Fort Worth

BOOKS RECEIVED FOR REVIEW

Systems and Roots. By J. M. Thomas, William Byrd Press, Richmond, 1962, x+123 pages,$5.00.

Modern Algebra, Second Course. By R. E. Johnson, L. L. Lendsey, W. E. Slesnick, and G. E. Bates, Addison-Wesley, Reading, Mass., 1962, xii+594 pages, $5.44.

Numerical Mathematical Analysis, Fifth edition. By J. B. Scarborough, Johns Hopkins Press, Baltimore, 1962, xxi+594 pages, $7.00.

Russian-English Mathematical Dictionary. By L. M. Milne-Thomson, C.B.E., The University of Wisconsin Press, Madison, 1962, xiii+191 pages, $5.00.

Modern Geometry, Its Structure and Function. By K. B. Henderson, R. E. Pingry and G. A. Robinson, McGraw-Hill, New York, 1962, xiv+561 pages, $5.36.

Retracing Elementary Mathematics. By Leon Henkin, W. N. Smith, V. J. Varineau and M. J. Walsh, Macmillan, New York, 1962, xviii+418 pages, $6.50.

Invitation to Mathematics. By William H. Glenn and Donovan A. Johnson, Doubleday, New York, 1962, 373 pages, $4.95.

Basic Mathematics Review. By James A. Cooley, Macmillan, New York, 1962, iii+279 pages, $3.50 (paper).

Elements of Finite Mathematics. By Francis J. Scheid, Addison-Wesley, Reading, Mass., 1962, vii+279 pages, $6.75.

Finite Mathematics with Business Applications. By John G. Kemeny, Arthur Schleifer, Jr., J. Laurie Snell and Gerald L. Thompson, Prentice-Hall, Englewood Cliffs, N. J., 1962, xii+482 pages, $7.95.

Introductory College Mathematics, Third edition. By W. E. Milne and D. R. Davis, Ginn, Boston, 1962, xii+579 pages, $7.50.

First Year Mathematics for Colleges, Second edition. By Paul R. Rider, Macmillan, New York, 1962, xvi+667 pages, $7.50.

Basic College Algebra. By J. D. Mancill and M. O. Gonzalez, Allyn and Bacon, Boston, 1962, xi+458 pages, $6.75.

Elements of Algebra. By J. H. Banks and F. L. Wren, Allyn and Bacon, Boston, 1962, xi+514 pages, $6.50.

Algebra II. By C. F. Brumfiel, R. E. Eicholz, and M. E. Shanks, Addison-Wesley, Reading, Mass., 1962, xiii+466 pages, $5.28.

Algebra and Trigonometry. By P. K. Rees and F. W. Sparks, McGraw-Hill, New York, 1962, v+403 pages, $6.95.

Functional Trigonometry. By A. P. Hillman and G. L. Alexanderson, Allyn and Bacon, Boston, 1961, xiii+327 pages, $5.95.

Understanding Mathematics with Visual Aids. By L. A. Kenna, Littlefield, Adams, Paterson, N. J., 1962, xi+174 pages, $1.75 (paper).

So You're Going to Be a Teacher. By Robert L. Filbin and Stefan Vogel, Barron's Educational Series, Great Neck, N. Y., 1962, iii+141 pages, $2.95 (cloth), $1.25 (paper).

Mathematics for Business and Economics. By Robert Cissell and Thomas J. Bruggeman, Houghton Mifflin, Boston, 1962, ix+229 pages, $4.75.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 *North Vermont Avenue, Los Angeles* 29, *California.*

PROPOSALS

495. *Proposed by Maxey Brooke, Sweeney, Texas.*

"I just surveyed a peculiar field," said an engineer, "it is the smallest right triangle with integral sides whose perimeter is a cube." What are the dimensions of the field?

496. *Proposed by C. W. Trigg, Los Angeles City College.*

Solve this mixed cryptarithm in which each letter uniquely represents a digit in the decimal system:

```
      E R E
      * * *
    -------
    * * 2 1
  L O N G
  * * *
  ---------
* R * * * *
```

497. *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

Show that

$$4\sum_{r=0}^{n} r^3\binom{n}{r}^p = 6n\sum_{r=0}^{n} r^2\binom{n}{r}^p - n^3\sum_{r=0}^{n}\binom{n}{r}^p.$$

498. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If m and n are integers and δ, D are their g.c.d. and l.c.m. respectively, and $d(n)$ denotes the number of divisors of n, $\phi(n)$ being the Euler function, prove that:

(1) $$d(m)d(n) = d(\delta)d(D)$$

(2) $$\phi(m)\phi(n) = \phi(\delta)\phi(D)$$

499. *Proposed by N. C. Perry, Auburn University, Alabama.*

Prove that if n points are chosen at random on a circle, the probability that they all lie on the same semi-circle is $n/2^{n-1}$.

500. *Proposed by Joseph Hammer, University of Sidney, Australia.*

If x_1', x_2', x_3' are the images of the vectors x_1, x_2, x_3 of 3-dimensional space under the linear transformation

$x_1' = a_1x_1 + a_2x_2 + a_3x_3$

$x_2' = b_1x_1 + b_2x_2 + b_3x_3$

$x_3' = c_1x_1 + c_2x_2 + c_3x_3$ where the values of $a_1, a_2, \cdots, c_3$ are the integers from 1 to 9 taken in some order, which ordering will make the volume of the parallelopiped defined by x_1', x_2', x_3' equal to the volume of the one defined by x_1, x_2, x_3?

501. *Proposed by Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

In the usual triangle ABC, with corresponding opposite sides a, b, c, denote the median, internal bisector, and symmedian issued from A by t_m, t_a, and t_s respectively. Let t be the join of A to the point P on BC which bisects the perimeter. Show that:

(1) $t_s = Ht_m/(2M-H)$ where M, H are the arithmetic and harmonic means of b and c

(2) $t_a = H \cos 1/2A$

(3) $t^2 = s^2\left(1 - \dfrac{r}{R}\sec^2 \dfrac{1}{2}A\right)$ where s is the semi-perimeter, r and R the inradius and circumradius of the given triangle.

SOLUTIONS

Late Solutions

459. *Gregory Lutz (age 13), Longview, Washington.*

467, 470. *Josef Andersson, Vaxholm, Sweden.*

473. *Joe Scanio, Harvard University.*

The Irreducibility of Certain Polynomials

474. [March 1962]. *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

The three polynomials $x-x$, x^2+y^2-2xy, and $x^3+y^3+z^3-3xyz$ can each be factored into real polynomials. Which if any of the higher order analogous polynomials

$$\sum_{r=1}^{n} x_r^n - nx_1x_2 \cdots x_n$$

are reducible?

Solution by J. A. Tyrrell, King's College, London.

None of the higher order polynomials are reducible (into either real or complex factors.) To see this, observe that a factorization of

$$x_1^n + x_2^n + x_3^n \tag{1}$$

could be obtained from any factorization of the given polynomial (for $n \geq 4$) merely by setting $x_4, \cdots, x_n$ equal to zero. As (1) is well-known to be irreducible (for all positive integers n) our assertion follows. The following proof that (1) is irreducible may be of interest. (The impossibility of *linear* factors is trivial to demonstrate.) To prove the more general assertion, interpret x_1, x_2, x_3 as the homogeneous coordinates of a point in a projective plane; the vanishing of (1) then represents a plane curve of order n and, since the partial derivatives of (1) with respect to the x_1 do not vanish simultaneously at any point of the plane, this curve is non-singular (i.e. it has no multiple points). If the expression (1) were factorable, the curve would be reducible and then would necessarily possess multiple points (at the points of intersection of any two components). It follows that the expression (1) is irreducible. (*Note*: the geometrical facts used here may be looked up in any elementary treatise on Higher Plane Curves.)

Also solved by L. Carlitz, Duke University.

The Unit Repetend

475. [March 1962]. *Proposed by Guy G. Becknell, Tampa, Florida.*

Find a unique set of six integers which form a group having the peculiar property that the product of any five of them is one or more periods of the unit repetend number of the remaining one. For example

$$\frac{1}{41} = .\overline{02439}.$$

Hence, the unit repetend number for 41 is 02439, equivalent to 2439 numerically.

Solution by C. W. Trigg, Los Angeles City College.

If $a_2a_3a_4a_5a_6$ is the unit repetend number of a_1 then $a_1a_2a_3a_4a_5a_6$ consists of a series of 9's. Now $99=(9)(11)$, $999=(3)(9)(37)$, $9999=(9)(11)(101)$, $99999=(9)(41)(271)$, and $999999=(3)(7)(9)(11)(13)(37)$. The last set is the *smallest* set of six distinct integers having the stated property. Indeed

$1/3 = 0.333333$	$1/11 = 0.090909$
$1/7 = 0.142857$	$1/13 = 0.076923$
$1/9 = 0.111111$	$1/37 = 0.027027$

However, this set of six distinct integers is *not* unique. Solutions result from any series of $6k$ 9's. For example: $999999999999=(3)(7)(9)(11)(13)(37)(101)(9901)$. This set of eight distinct integers can be compressed into sets of six distinct integers in $C(8, 3)+[C(8, 2)][C(6, 2)]+C(6, 3)$ or 496 ways, thus obtaining such typical solutions as:

(3)(7)(9), 11, 13, 37, 101, 9901;
(3)(7), (9)(11), 13, 37, 101, 9901;
and (3)(7), (3)(11), (3)(13), 37, 101, 9901.

Solved also by Brother Alfred, St. Mary's College, California; Maxey Brooke, *Sweeney, Texas; Gilbert Labelle, Collège de Longueuil, Canada; C. F. Pinzka, University of Cincinnati; and the proposer.*

Perimeter and Area Bisector

476. [March 1962]. *Proposed by Kaidy Tan, Fukien Normal College, China.*

Draw a straight line bisecting the perimeter and area of a given quadrilateral.

Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.

We consider two cases according as the line intersects two adjacent or two opposite sides. Either case includes the case in which the line contained a vertex.

I. The line intersects two adjacent sides (fig. 1.)

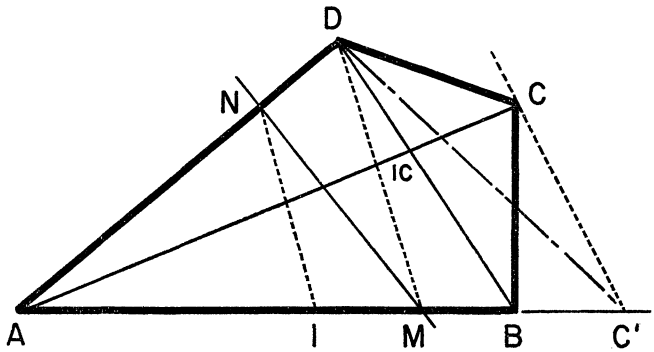


FIG. 1

(a) The line bisects the area:

Drawing $CC' \| BD$ we have area $ABCD = ABD + BCD = ABD + BC'D = AC'D$. Let I be the midpoint of AC', and M be a point on AB. Drawing $IN \| MD$ we have $AMN = AIN + IMN = AIN + IDN = AID = \frac{1}{2}AC'D = \frac{1}{2}ABCD$. Hence MN so constructed bisects the area. The constructions give:

Let $AM = m$, $AN = n$, then

$$AC'/AC = AB/AK = a/\alpha, \qquad AC' = a \cdot AC/\alpha = pa/\alpha.$$

$$AM/AD = AI/AN, \qquad mn = AI \cdot AD = \tfrac{1}{2}AC' \cdot d = pad/2\alpha.$$

$$2mn = pad/\alpha.$$

(b) The line bisects the perimeter:

$$m + n = AM + AN = \tfrac{1}{2}(a + b + c + d).$$

Therefore m, n determining the line MN are the roots of the quadratic equation:

$$2x^2 - (a + b + c + d)x + pad/\alpha = 0$$

For the existence of MN we have the conditions:

(1) $m \leq b$, $n \leq d$ or $a+b+c+d = 2m+2n \leq 2b+2d$ or $b+c \leq a+d$

(2) $pad/\alpha=2mn\leq 2ad$ or $p\leq 2\alpha$, $\alpha+\gamma 2\alpha$, $\alpha\geq\gamma$.

(3) $\Delta=(a+b+c+d)^2-8pad/\alpha\geq 0$.

II. The line intersects two opposite sides.

(a) The line bisects the area:

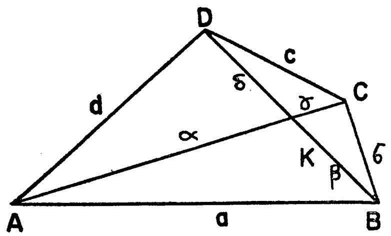


FIG. 2

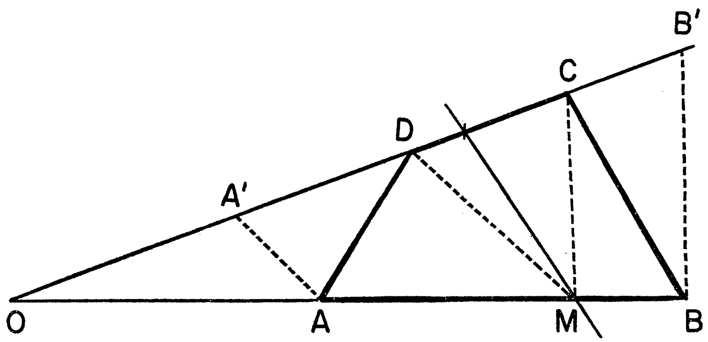


FIG. 3

Let M be any point on AB (fig. 3). Draw $BB'\|MC$, and $AA'\|MD$. Then: $ABCD = AMD + MCD + BCM = A'MD + MCD + B'CM = MDA' + MCD + MB'C = MB'A'$. If N is the midpoint of $A'B'$, the lkne MN will bisect $MB'A'=ABCD$. Let $OM=m$, $ON=n$, $OA=a'$, $OB=b'$, $OC=c'$, $OD=d'$, $OA'=a''$, $OB'=b''$. Then from the constructions:

$$m/c' = b'/b'', \qquad m/d' = a'/a''$$

$$2n = a'' + b'' = a'd'/m + b'c'/m = (a'd' + b'c')/m$$

$$2mn = a'd' + b'c'.$$

(b) The line bisects the perimeter:

$$MA + AD + DN = MB + BC + CN$$

$$(m - a') - d + (n - d') = (b' - m) + b + (c' - n)$$

$$2(m + n) = (a' + b' + c' + d') + (b - d.)$$

Therefore m, n determining the line MN are the roots of the quadratic equation

$$2x^2 - (a' + b' + c' + d' + b - d)x + (a'd' + b'c') = 0.$$

The existence of MN is given by $a' \leq m \leq b'$, $d' \leq m \leq c'$ which yield $a+c+b \geq d$ and $a+c+d \geq b$ which are always true.

A Game of Sudden Death

477. [March 1962]. *Proposed by Maxey Brooke, Sweeney, Texas.*

Before they began, Brown had twice as much money as Calhoun while Adams had three times as much. The game was called Sudden Death. Two cards are dealt to each player. He puts into the pot an amount equal to the product of those cards (jacks, queens, and kings count as 11, 12, and 13). A third card is dealt and the one receiving the low third card wins the pot. During the evening there were losses and wins. But on the last hand, Adams put one-half the money he had in the pot, Brown put in one-third of his money, and Calhoun put in one-sixth. Each player drew a three as the last card and they split the pot equally. On counting up, each player found that he had the same amount that he began with. How much was it?

Solution by Herbert R. Leifer, Pittsburgh, Pennsylvania.

Let A, B, C, be the amounts Adams, Brown, and Calhoun respectively had at the start of the game, and A', B', C', the amounts they had respectively at the start of the last hand. At the end of the game, $A'/2+(3A'+2B'+C')/18=A$, $2B'/3+(3A'+2B'+C')/18=B$, $5C'/6+(3A'+2B'+C')/18=C$, and $B=2C$, $A=3C$. From which it is readily found that $A'=198C/47$. Since the amounts are integral, let $C=47r$, r a positive integer, then $A=141r$, $B=94r$, $A'=198r$, $B'=78r$, and $C'=6r$. Since what is put into the pot is the product of two dealt cards, the highest product possible is 169, and thus $r=1$. (On the last hand Adams had a Jack and 9 dealt to him, Brown a deuce and King, Calhoun two aces.) Thus they started with 47, 94, and 141 dollars, respectively.

Also solved by Brother U. Alfred, St. Mary's College, California; Merrill Barneby, University of North Dakota; Daniel I. A. Cohen, Central High School, Philadelphia, Pennsylvania; C. F. Pinzka, University of Cincinnati; Jean Richard, Lachenaie, P.Q., Canada; David L. Silvermann, Beverly Hills, California; C. W. Trigg, Los Angeles City College; and the proposer.

Palindromic Numbers

478. [March 1962]. *Proposed by C. W. Trigg, Los Angeles City College.*

a) Identify the four five-digit palindromic numbers whose squares are composed of distinct digits.

b) The square of a permutation of one of these palindromes also has distinct digits. Find it and show that it is the only one.

Solution by C. F. Pinzka, University of Cincinnati.

(a) Let $N=(abcba)$ be the palindrome. If N^2 has 9 digits, then $102{,}345{,}678 < N^2 < 987{,}654{,}321$ or $10201 \leq N \leq 31513$, a total of 214 possibilities. Many of these can be eliminated on obvious considerations. For example, the squares of 10201 and 14141 begin and end with 1. Checking the remaining ones with a table of squares, we find only $28582^2 = 816{,}930{,}724$. If N^2 has all 10 digits, then $N^2 \equiv 0 \pmod 9$ or $N \equiv 0 \pmod 3$. This implies $a+b \equiv c \pmod 9$. Since now $32223 \leq N \leq 99099$, there are several hundred cases to check. Proceeding as before, we find:

$$35853^2 = 1{,}285{,}437{,}609$$
$$84648^2 = 7{,}165{,}283{,}904$$
$$97779^2 = 9{,}560{,}732{,}841.$$

(b) The solutions found in (a) have a total of 100 permutations. Checking these, we find

$$85353^2 = 7{,}285{,}134{,}609$$

as the only permutation with the required property.

Also solved by Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; Daniel I. A. Cohen, Central High School, Philadelphia, Pennsylvania; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Gilbert Labelle, Collège de Longueuil, Canada; and the proposer.

Inscribed Circles

479 [March 1962]. *Proposed by M. N. Gopalan, Mysore City, India.*

ABC is a right triangle with right angle at C. CD is drawn perpendicular to AB. r_1 and r_2 are the in-radii of the triangles formed and r is the in-radius of triangle ABC. Prove that $r_1^2+r_2^2=r^2$.

I. *Solution by Hazel S. Wilson, Jacksonville University, Florida.*

Set up a coordinate system with C at the origin. Let the coordinates of A and B be $(a, 0)$ and $(0, b)$, respectively. Then the sides of triangle ABC are a, b, and c, where $c=a^2+b^2$. The equation of AB is $bx+ay=ab$, and of CD is $ax-by=0$. Solving these equations, gives as the coordinates of D,

$$x' = ab^2/c^2, \qquad y' = a^2b/c^2.$$

Using the distance formula,

$$BD = b^2/c, \qquad AD = a^2/c, \qquad CD = ab/c.$$

The in-radius is $r=K/s$, where K is the area of the triangle and s is the semiperimeter.

$$K = ab/2, \qquad s = (a+b+c)/2$$
$$K_1 = 1/2BD\cdot CD = (ab^3)/(2c^2), \qquad s_1 = b(a+b+c)/(2c) = bs/2c$$
$$K_2 = AD\cdot CD = (a^3b)/(2c^2), \qquad s_2 = as/2c$$
$$r = ab/2s, \qquad r_1 = ab^2/2cs, \qquad r_2 = a^2b/2cs$$
$$r_1^2 + r_2^2 = (a^2b^4 + a^4b^2)/4c^2s^2a^2b^2(a^2+b^2)/4c^2s^2 = a^2b^2/4s^2 = r^2$$

II. *Solution by Leon Bankoff, Los Angeles, California.*

Let r_1 denote the inradius of triangle ADC, and r_2 that of triangle CDB. In the similar right triangles ABC, CAD, and BCD, the ratios of inradius to hypotenuse are equal. So $r_1 = r(AC/AB)$; $r_2 = r(CB/AB)$ and $r_1^2 + r_2^2 = r^2(AC^2+CB^2)/AB^2 = r^2$. This property was listed along with 23 others in a note following the solution of problem E 1197 in the *American Mathematical Monthly*, Sept. 1956, pp. 493–5.

Also solved by Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Joseph B. Bohac, St. Louis, Missouri; Maurice Brisebois, Université de Sherbrooke, Canada; Daniel I. A. Cohen, Central High School, Philadelphia, Pennsylvania; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Robert P. Goldberg, Cambridge, Massachusetts; Ralph Greenberg, Philadelphia, Pennsylvania; Roger D. H. Jones, College of William and Mary; Darryl E. Kuhns, IBM, Los Angeles; Gilbert Labelle, Collège de Longueuil, Canada; Andrzej Makowski, Warsaw, Poland; C. F. Pinzka, University of Cincinnati; C. W. Trigg, Los Angeles City College; Brother Louis Zinkel, Marist College, New York; and the proposer.

480. [March 1962]. *Proposed by Gilbert Labelle, Collège de Longueuil, Canada.*

Prove that 2^p-1 is not a prime number if the prime p is of the form $4m-1$ and $2p+1$ is also a prime number.

Solution by Sidney Kravitz, Dover, New Jersey.

This theorem was proved by Euler who showed that 2^p-1 is composite, under the given conditions, because it is divisible by $2p+1$. The proof is as follows: if a prime r is of the form $8m\pm1$ then 2 is a quadratic residue and $2^{\frac{r-1}{2}}\equiv 1 \bmod (r)$. In this case $2p+1$ is of the form $8m-1$ so that $2^p\equiv 1 \bmod (2p+1)$.

Also solved by Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; J. L. Brown, Jr., Pennsylvania State University; C. F. Pinzka, University of Cincinnati; Philip Smedley, McMaster University; Stephen Ullom, American University; and the proposer.

Several solvers and the proposer pointed out the necessity for $m>1$. Smedley located the proof in *The Theory of Numbers*, Hardy and Wright, Fourth Edition.

COMMENTS ON SOLUTIONS

440. [March and November 1961]. *Comment by Huseyin Demir, Middle East Technical University, Akara, Turkey.*

The number N given in the statement is correct. N denotes the number of small circles contained entirely by the larger circle (tangency being included). The number N offered by A. Sutcliffe includes also the partly contained circles and therefore both numbers are correct.

445. [March and November 1961]. *Comment by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

The proof needs a little modification. Read: Let the orthogonal projection of A, B and P on the line OM be A', B' and P'. In the remaining part of the proof all letters P are to be replaced by P' and in conclusion we have

$$\frac{MA^2 - k}{MB^2 - k} = \frac{2OM \cdot A'P'}{2OM \cdot B'P'} = \frac{A'P'}{B'P'} = \frac{PA}{PB}.$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 303. If every rth term is removed from the series $1-1/2+1/3-1/4+\cdots$, find the resulting sum. [*Submitted by M. S. Klamkin.*]

Q 304. Prove that between any two positive reals there exists a perfect square rational. [*Submitted by Norman Schaumberger and Erwin Just.*]

Q 305. If the rational roots of $x^3+ax+b=0$ are m, n, p, then the roots of $my^2+ny+p=0$ are also rational. [*Submitted by C. W. Trigg.*]

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase, or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 54. Find integers x, y, and z such that for $n>2$

$$x^{-n} + y^{-n} = z^{-n}$$

[*Submitted by David L. Silverman.*]

T 55. Find all the solutions to the Diophantine equation

$$y = \frac{x^{11}}{11} + \frac{x^{10}}{2} + \frac{5x^9}{2} - x^7 + x^5 - \frac{x^3}{2} + \frac{5x}{66}.$$

[*Submitted by M. S. Klamkin.*]

T 56. Describe the family of curves whose equation is $(x+y+1)^2 = 3(x+y-xy-a^2)$ where x and y are real and a is a real parameter. [*Submitted by M. S. Klamkin.*]

T 57. Using only the elementary operations along with decimal, power, root and factorial notations, express 31 in terms of four 4's. [*Submitted by Marlow Sholander.*]

Answers and Solutions on p. 272.

ANSWERS TO QUICKIES

A 303. If r is even the resulting series obviously diverges. If r is odd, then

$$S_{rn-r} = \left(1 - 1/2 + 1/3 - \cdots \frac{1}{rn}\right) - \frac{1}{r}\left(1 - 1/2 + 1/3 - \cdots \frac{1}{n}\right)$$

$$S = \lim_{n\to\infty} S_{rn-r} = \left(1 - \frac{1}{r}\right) \ln 2.$$

A 304. Assume $a<b$. Then there exists a rational r such that $\sqrt{a}<r<\sqrt{b}$, which implies that $a<r^2<b$ as required.

A 305. The sum of the roots, $m+n+p=0$, so the roots of $my^2-(m+p)y+p=0$ or $(y-1)(my-p)=0$ are 1 and p/m.

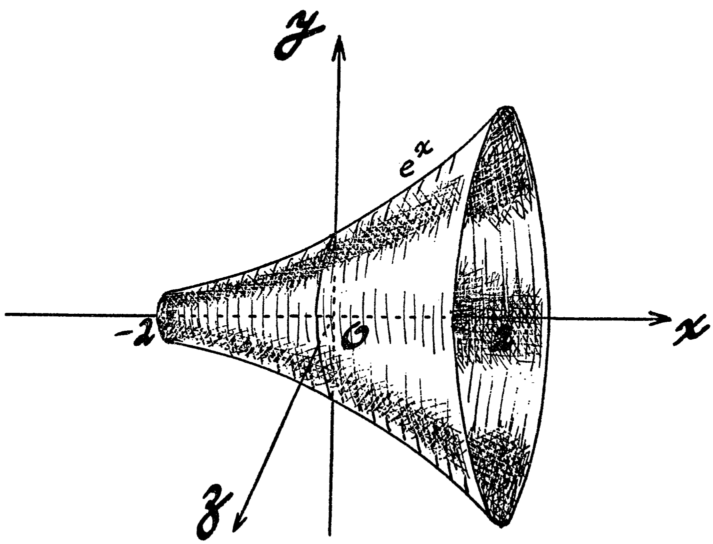
SOLUTIONS OF TRICKIES

S 54. Multiply through by $(xyz)^n$ gives $(yz)^n+(xz)^n=(xy)^n$, so that finding integers satisfying $x^{-n}+y^{-n}=z^{-n}$ requires finding a counterexample to Fermat's Last Theorem.

S 55. Since $y=\sum_{r=1}^{x} r^{10}$, y is integral for all integers x.

S 56. The equation can be rewritten as $(x-1)^2+(y-1)^2+(x-y)^2=-6a^2$. Whence the family consists of the single point (1, 1).

S 57. It may be written $31 = {}^{.4}\sqrt{4} - 4/4$.



The surface obtained by rotating $y=e^x e^x$, $-2\le x\le 2$ about the x-axis.

THE MATHEMATICAL ASSOCIATION OF AMERICA

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Further information about the Association, its publications and its activities may be obtained by writing to:

HARRY M. GEHMAN, *Executive Director*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

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